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# Notes connected with algebraic functions

## 1 Minimal polynomial algebraic functions having a singular point with defined type

The simplest conditions for a singular point at $(z_0, w_0)$ are either

$$\frac{\partial P}{\partial z}(z_0, w_0) = 0 \tag{1}$$

or

$$\frac{\partial P}{\partial w}(z_0, w_0) = 0. \tag{2}$$

These conditions can be extended by adding to them

$$\left.\frac{\partial^{s+t} P}{\partial z^s \partial w^t}\right|_{z_0, w_0} = 0 \tag{3}$$

for any $(s,t) \in S$ where the set $S$ satisfies

$$\begin{array}{l}(s,t) \in S \text{ implies both } (s-1,t) \in S \text{ (provided } s-1 \geq 0) \\ \text{and } (s,t-1) \in S \text{ (provided } t-1 \geq 0).\end{array} \tag{4}$$

These conditions ensure that a derivative is not equated to zero when a more significant derivative (corresponding to a more significant i.e. lower order term in the Taylor series) at the same point is non-zero. There is another set of derivatives that dominate (are of lower order than) any other derivative not required to be zero. These are the derivatives in (3) for all $(s,t) \in T$ where $T$ is such that

$$\begin{array}{l}\text{for each } (s,t) \text{ such that } s \geq 0 \text{ and } t \geq 0 \text{ and } (s,t) \notin S \text{ then} \\ \text{for at least one } (k,l) \in T, s \geq k \text{ and } t \geq l.\end{array} \tag{5}$$

Then $S$ is uniquely determined by $T$ as the set $(s,t)$ such that for all $(k,l) \in T, s < k$ or $t < l$ i.e.

$$S = \{(s,t) : \forall (k,l) \in T (0 \leq s < k \text{ or } 0 \leq t < l)\} \tag{6}$$

The set $T$ is also unique once $S$ is determined. This is because each member of $T$ imposes a condition on $S$ and none of these conditions can be deduced from the others (if that happened the deduced ones would be removed from $T$), so if $(s_1, t_1) \in T$ this implies that $0 \leq s < s_1$ or $0 \leq t < t_1$ so this condition must therefore be in any alternative $T_1$ to $T$ that has the same effect (same $S$) or deduced from it, but the latter is ruled out because any $T$ is defined to be minimal as described above. This shows that any member of $T$ is in $T_1$ and vice versa therefore $T = T_1$ so $T$ is unique.

For finding the derivatives of the polynomial the following result is needed:

$$\frac{\partial^k z^s}{\partial z^k} = \frac{s! z^{s-k}}{(s-k)!} \text{ if } k \leq s \text{ and } 0 \text{ otherwise} \tag{7}$$

so

$$\frac{\partial^{k+l}}{\partial z^k \partial w^l}(z^s w^t) = \begin{cases} \dfrac{s!t! z^{s-k} w^{t-l}}{(s-k)!(t-l)!} & \text{if } k \leq s \text{ and } l \leq t \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

Now consider what terms need to be included in the polynomial $P(z, w) = \sum\sum a_{st} z^s w^t = 0$ that represents the function $w(z)$. If $(s_0, t_0) \in T$ and $a_{s_0 t_0} = 0$ then the required non-zero value for $\left.\frac{\partial^{s_0+t_0} P}{\partial z^{s_0} \partial w^{t_0}}\right|_{z_0, w_0}$ can only come from term(s) $a_{st} z^s w^t$ where $s \geq s_0$ and $t \geq t_0$. Therefore the simplest i.e. lowest order choice of polynomial (the minimal polynomial as in this section heading) is when $a_{st} \neq 0$ for all $(s, t) \in T$ and $a_{st} = 0$ for all non-negative integer pairs $(s, t) \notin S \cup T$. This gives

$$P(z, w) = \sum_{(k,l) \in S \cup T} \sum a_{kl} z^k w^l = 0. \tag{9}$$

There are presumably interesting cases with polynomials not satisfying (9) when more than one singular point is expected, but the following analysis concerns only cases when (9) holds. The following system

$$\left.\frac{\partial^{s+t} P}{\partial z^s \partial w^t}\right|_{z_0, w_0} = 0 \text{ for all } (s, t) \in S \tag{10}$$

involving the parameters

$$\left.\frac{\partial^{s+t} P}{\partial z^s \partial w^t}\right|_{z_0, w_0} \text{ for } (s, t) \in T \tag{11}$$

must be solved for the $a_{st}$ for $(s, t) \in S \cup T$. Sustituting (9) into $\frac{\partial^{s+t} P}{\partial z^s \partial w^t}$ and using (8) gives

$$\left.\frac{\partial^{s+t} P}{\partial z^s \partial w^t}\right|_{z_0, w_0} = \sum_{\substack{(k,l) \in S \cup T \\ k \geq s,\ l \geq t}} \sum a_{kl} \left( \frac{k! l! z_0^{k-s} w_0^{l-t}}{(k-s)!(l-t)!} \right) \text{ for all } (s, t) \in S \cup T. \tag{12}$$

For $(s, t) \in T$ there is just a single term in the sum. It has $k = s$ and $l = t$ so

$$a_{st} = \frac{1}{s!t!} \left.\frac{\partial^{s+t} P}{\partial z^s \partial w^t}\right|_{z_0, w_0} \text{ for all } (s, t) \in T. \tag{13}$$

Equation (12) relates $a_{st}$ to only other values $a_{kl}$ with $k \geq s$ and $l \geq t$. If the latter have already been found, $a_{st}$ can be determined. The latter themselves can be solved from other members of (12) likewise. Therefore if for each $(s,t) \in S \cup T$

$$p = \#\{(k,l) \in S \cup T : k \geq s \text{ and } l \geq t\} \tag{14}$$

is introduced, every element $a_{st}$ can be solved for in terms of other $a_{kl}$ with a smaller value of $p$. Therefore (12) must be solved for the $a_{st}$ in any order in which $p$ is non-decreasing. This shows that the $a_{st}$ are uniquely determined from (12).

The result of this is

$$P(z,w) = \sum_{(s,t)\in T} \frac{(z-z_0)^s}{s!} \frac{(w-w_0)^t}{t!} \left.\frac{\partial^{s+t} P}{\partial z^s \partial w^t}\right|_{z_0,w_0} \tag{15}$$

because this is the Taylor series expansion of $P$, using (10), about $(z_0, w_0)$ truncated so that no terms with $(z-z_0)^k(w-w_0)^l$ such that $k > s$ or $l > t$ for any $(s,t) \in T$ contribute in accordance with (9). To consider singular points the following derivative is also needed

$$\frac{\partial P}{\partial z} = \sum_{(s,t)\in T,\ s>0} \frac{(z-z_0)^{s-1}}{(s-1)!} \frac{(w-w_0)^t}{t!} \left.\frac{\partial^{s+t} P}{\partial z^s \partial w^t}\right|_{z_0,w_0}. \tag{16}$$

The following notation will be used for $(s,t) \in T$ where $\#(T) = k+1$,

$$T = \{(s_0,t_0), (s_1,t_1), \ldots (s_k,t_k)\} \tag{17}$$

where the $s's$ and $t's$ are non-negative integers and the $s's$ increase with the subscript and $t's$ decrease with the subscript i.e.

$$q < r \text{ implies } s_q < s_r \text{ and } t_q > t_r, \text{ and } s_0 = 0 \text{ and } t_k = 0. \tag{18}$$

It is also convenient to indroduce

$$\Sigma = \{s_0, s_1 \ldots s_k\}. \tag{19}$$

Therefore from (18), $0 \in \Sigma$ and $s_k \in \Sigma$ later named $n$ in the EA.

To answer the question of whether (15) has any singular points other than $(z_0, w_0)$, the Euclidean algorithm will be used with (15) and (16), regarding these as polynomials in $z - z_0$. At the first step (15) is divided by (16) just considering the leading powers of $z - z_0$. The first quotient and remainder are obtained removing an overall factor $w - w_0$, then (16) takes the place of (15) and the the remainder takes the place of (16) and this is repeated until 0 is obtained. The previous remainder is the necessary and sufficient condition under which both (15) and (16) hold i.e. one of the conditions for a singular point other than when $w = w_0$.

## 1.1 Study of the general case where $T = \{(0,2),(1,1),(3,0)\}$

### 1.1.1 Finding the singular points

This implies (15) takes the form

$$P(z,w) = \frac{(w-w_0)^2}{2} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + (z-z_0)(w-w_0) \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} + \frac{(z-z_0)^3}{6} \left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} \tag{20}$$

from which follows

$$\frac{\partial P}{\partial z} = (w-w_0) \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} + \frac{(z-z_0)^2}{2} \left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0}. \tag{21}$$

Both the (20) and (21) individually equated to 0 show that if $w = w_0$ then $z = z_0$ and conversly because the partial derivatives in (15) and (20) are all non-zero. Eliminating the cubic term in $z - z_0$ shows that because of (21) equated to 0, (20) can be replaced by (22)

$$P - \frac{(z-z_0)}{3}\frac{\partial P}{\partial z} = \frac{(w-w_0)^2}{2} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + \frac{2}{3}(z-z_0)(w-w_0) \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} \tag{22}$$

When trying to find common solutions to (21) and (22) equated to 0 with $z \neq z_0$ (and therefore $w \neq w_0$), $w - w_0$ can be factored out giving

$$\frac{(w-w_0)}{2} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + \frac{2}{3}(z-z_0) \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} = 0 \tag{23}$$

so common solutions to (23) and (21) must be found. Eliminating the highest powers of $z - z_0$ and again taking out the $w - w_0$ factor gives

$$z - z_0 = \frac{8}{3} \frac{\left.\frac{\partial^2 P}{\partial z \partial w}^2\right|_{z_0,w_0}}{\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}} \tag{24}$$

This combined with (23) gives

$$w - w_0 = -\frac{32}{9} \frac{\left.\frac{\partial^2 P}{\partial z \partial w}^3\right|_{z_0,w_0}}{\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} \left.\frac{\partial^2 P}{\partial w^2}^2\right|_{z_0,w_0}} \tag{25}$$

Checking this solution ((24) and (25) which will be denoted by $(z_1, w_1)$) by substituting it back shows that (20) and (21) are satisfied and the assumption that $w \neq w_0$ is eliminating the other solution $z = z_0, w = w_0$. In a similar way the common solution of $P = \frac{\partial P}{\partial w} = 0$ other than $z = z_0, w = w_0$ was obtained by treating $w - w_0$ as the variable giving

$$z - z_0 = 3 \frac{\left.\frac{\partial^2 P}{\partial z \partial w}^2\right|_{z_0,w_0}}{\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}} \tag{26}$$

$$w - w_0 = -3 \frac{\left.\frac{\partial^2 P}{\partial z \partial w}^3\right|_{z_0,w_0}}{\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} \left.\frac{\partial^2 P}{\partial w^2}^2\right|_{z_0,w_0}} \tag{27}$$

which will be denoted by $(z_2, w_2)$. The similarity of these results is surprising and gives

$$\begin{array}{l} z_1 - z_0 = \frac{2^3}{3^2}(z_2 - z_0) \\ w_1 - w_0 = \frac{2^5}{3^3}(w_2 - w_0) \end{array} . \tag{28}$$

The single singular point expected is obtained if $\left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} = 0$. This implies that a non-zero value of the mixed second derivative gives rise to a pair of singular points. In many cases algebraic functions have a singular point at $\infty$ that is easily overlooked. The simplest example is $f(z) = 1/z$ that is determined by $P = zw - 1 = 0$. The singular points are given by $\frac{\partial P}{\partial z} = 0$ i.e. $w = 0$ or $z = \infty$, and $\frac{\partial P}{\partial w} = 0$ i.e. $z = 0$.

### 1.1.2 Characterising these singular points

In order to determine the sets $S$ (4) and $T$ and the leading terms in the expansion of $w$ in terms of $z$ for each singular point, derivatives will be needed evaluated at $(z_1, w_1)$ and $(z_2, w_2)$ as well as at $(z_0, w_0)$. Starting with the main singular point $(z_0, w_0)$, an expansion of the form

$$w - w_0 = \sum_{i=0}^{\infty} a_i (z - z_0)^{r_i} \ldots \tag{29}$$

will be sought. The terms will be in decreasing order of significance i.e. $r_i$ increases as $i$ increases, and $r_0 > 0$.

For $(z_1, w_1)$ from (21), the terms cancel giving

$$\left.\frac{\partial P}{\partial z}\right|_{z_1,w_1} = 0 \tag{30}$$

and

$$\frac{\partial^2 P}{\partial z \partial w} = \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} \neq 0 \tag{31}$$

and

$$\frac{\partial^2 P}{\partial z^2} = (z - z_0) \left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} \neq 0, \tag{32}$$

and from (20)

$$\frac{\partial P}{\partial w} = (w - w_0) \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + (z - z_0) \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} \tag{33}$$

which at $(z_1, w_1)$ becomes

$$\frac{8}{9} \frac{\left.\frac{\partial^2 P}{\partial z \partial w}^3\right|_{z_0,w_0}}{\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}} \neq 0. \tag{34}$$

Therefore for $(z_1, w_1)$, $T = \{(0,1),(2,0)\}$ and $S = \{(0,0),(1,0)\}$ and the leading term in the expansion of $w - w_1$ is

$$w - w_1 = -\frac{9}{16} \frac{\left.\frac{\partial^3 P}{\partial z^3} \frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}}{\left.\frac{\partial^2 P}{\partial z \partial w}^2\right|_{z_0,w_0}} (z - z_1)^2. \tag{35}$$

For $(z_2, w_2)$, (33) implies

$$\frac{\partial^2 P}{\partial w^2} = \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} \neq 0, \tag{36}$$

and (31), and (21) at $(z_2, w_2)$ becomes

$$\frac{3}{2} \frac{\left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0}}{\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} \left.\frac{\partial^2 P}{\partial w^2}^2\right|_{z_0,w_0}} \neq 0 \tag{37}$$

so $S = \{(0,0),(0,1)\}$ so no extra derivatives for either singular point are zero.

Equation (20) is quadratic for $w$ and so can be solved for $w$ in terms of $z$. Write (20) as $A(w - w_0)^2 + B(w - w_0) + C = 0$ where $A = \frac{1}{2} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}$, $B = (z - z_0) \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0}$ and $C = \frac{(z-z_0)^3}{6} \left.\frac{\partial^2 P}{\partial z^3}\right|_{z_0,w_0}$. Writing (20) as a quadratic for $w$ alone shows that the descriminant is still $B^2 - 4AC$ and the solution is

$$w = w_0 + \frac{(z_0 - z) \left[ \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} + \left( \left.\frac{\partial^2 P}{\partial z \partial w}^2\right|_{z_0,w_0} - \frac{1}{3}(z - z_0) \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} \left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} \right)^{1/2} \right]}{\left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}}. \tag{38}$$

To check this, (24) and one of (38) is in agreement with (25), and (26) and (38) implies (27).

### 1.1.3 Developing the asymptotic series about a singular point

Next a series expansion of the form

$$w - w_0 = \sum_{i\geq 0}^{\infty} a_i (z - z_0)^{r_i} \tag{39}$$

will be developed for the singular point at $(z_0, w_0)$ where $a_i \neq 0$ and the terms are ordered so that $i < j$ implies $r_i < r_j$. These conditions insure that the terms are in decreasing order of significance for values of $z$ close to $z_0$. First substitute (39) into (20) giving

$$\begin{aligned} &\frac{1}{2}\sum_{i\geq 0}\sum_{j\geq 0} a_i a_j (z - z_0)^{r_i + r_j} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + \sum_{k\geq 0} a_k (z - z_0)^{r_k + 1} \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} \\ &+ \frac{(z - z_0)^3}{6} \left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} = 0 \end{aligned} \quad . \tag{40}$$

Therefore for each power of $z - z_0$ appearing in (40), the coefficients must total to zero. Although the $r_i$ have not yet been determined, this is possible with the following strategy. The most significant i.e. lowest order terms in each of the 3 expressions resulting from the 3 terms in (20) must each be cancelled by terms (not necessarily the lowest order) from another of those expressions. Working back from any term that cancels another to the most significant term in the set and which cancels it etc. the first terms to be considered are the most significant terms the whole of (40) which must cancel in pairs (or threes etc.). To do this, the powers of $z - z_0$ in the leading terms from the sets of terms derived from (20) are tested for equality in every pairwise combination and cancellation requires 2 or more of these to be equal and smaller than any of the other powers of $z - z_0$ in the set of other leading terms. This could happen in more than one way. If this condition can be satisfied, then there is an equation to be satisfied ensuring cancellation of these terms occurs. Then this is repeated with the remaining terms of (40) etc. to determine all the coefficients $a_i$ and $r_i$.

The general procedure is as follows, substitute (39) into (15) giving

$$P(z, w) = \sum_{(s,t)\in T} \frac{(z - z_0)^s}{s!} \frac{\left(\sum_{i\geq 0} a_i (z - z_0)^{r_i}\right)^t}{t!} \left.\frac{\partial^{s+t} P}{\partial z^s \partial w^t}\right|_{z_0,w_0} = 0 \tag{41}$$

For each term in the outer sum, the most significant is with the power of $(z - z_0)$ equal to $tr_0 + s$ and with coefficient

$$\frac{a_0^t}{s!t!} \left.\frac{\partial^{s+t} P}{\partial z^s \partial w^t}\right|_{z_0,w_0} . \tag{42}$$

The condition for cancellation of a pair of terms requires $t_0r_0 + s_0 = t_1r_0 + s_1$ giving $r_0 = \frac{s_1 - s_0}{t_0 - t_1}$ and the set of all lowest (most significant) powers of $z - z_0$ in the terms in the outer sum of (41) is then

$$\left\{ t_q \left( \frac{s_1 - s_0}{t_0 - t_1} \right) + s_q \right\} \tag{43}$$

for all $q \in \Sigma$. So the condition that none of these powers is more significant than those cancelled is

$$\min \left( \{ t_q r_0 + s_q \} \, q \in \Sigma \right) = t_0 r_0 + s_0. \tag{44}$$

The exponent $r_0$ must be chosen such that $(s_0, t_0)$ and $(s_1, t_1)$ satisfy this.

This has a pleasing geometrical interpretation. Consider lines of constant $tr_0 + s$ plotted on the graph of all the points in $T$. Then the function $tr_0 + s$ has the property that, because $r_0 > 0$, all points above this line have $tr_0 + s$ greater than its value on the line and (44) requires that all the points in $T$ are above the line joining $(s_0, t_0)$ and $(s_1, t_1)$. Thus all such pairs of points in $T$ can be read off directly from the plot of $T$ and the number and all the possible values of $r_0$ can be read off from the graph of the points of $T$ as the negative of the slopes of these lines (or their reciprocals depending on which way $T$ is plotted). This characterises the singular point as a multiple intersection of surfaces with different values of $r_0$.

Carrying this out for (40) shows that the leading terms have powers of $z - z_0$ equal to $2r_0$,$r_0 + 1$ and 3 and cancellation requires at least two of these to be equal, so equating all possible combinations gives

- $2r_0 = r_0 + 1 \Rightarrow r_0 = 1$, and both sides are equal to 2 and the other leading term has 3, so the leading terms can be cancelled.
- $2r_0 = 3 \Rightarrow r_0 = 3/2$. Both sides of the equation are 3 and the other leading term has index $r_0 + 1 = 5/2$ which is more significant so this more significant term could not be cancelled subsequently therefore this value of $r_0$ cannot be used.
- $r_0 + 1 = 3 \Rightarrow r_0 = 2$, and the other leading exponent is $2r_0 = 4$ which is less significant than these and could be cancelled subsequently.

For $r_0 = 1$, the cancellation of the leading terms requires

$$\frac{1}{2} a_0^2 \left. \frac{\partial^2 P}{\partial w^2} \right|_{z_0, w_0} + a_0 \left. \frac{\partial^2 P}{\partial z \partial w} \right|_{z_0, w_0} = 0 \tag{45}$$

from which the non-zero solution is

$$a_0 = \frac{-2 \left. \frac{\partial^2 P}{\partial z \partial w} \right|_{z_0, w_0}}{\left. \frac{\partial^2 P}{\partial w^2} \right|_{z_0, w_0}} \tag{46}$$

The remaining terms in (40) are

$$\frac{1}{2}\sum_{\substack{i\geq 0\\ i+j\geq 1}}\sum_{j\geq 0}a_i a_j (z-z_0)^{r_i+r_j}\left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}+\sum_{k\geq 1}a_k(z-z_0)^{r_k+1}\left.\frac{\partial^2 P}{\partial z\partial w}\right|_{z_0,w_0} \\ +\frac{(z-z_0)^3}{6}\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0}=0 \quad . \tag{47}$$

The most significant terms are now

$$\left\{a_0 a_1 (z-z_0)^{1+r_1}\left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}, a_1(z-z_0)^{1+r_1}\left.\frac{\partial^2 P}{\partial z\partial w}\right|_{z_0,w_0}, \frac{(z-z_0)^3}{6}\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0}\right\} \tag{48}$$

The next possible cancellation is for index $1+r_1$ and requires $a_0 a_1 \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + a_1 \left.\frac{\partial^2 P}{\partial z\partial w}\right|_{z_0,w_0} = 0$ which leads either to $a_1 = 0$ which is excluded or

$$a_0 \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + \left.\frac{\partial^2 P}{\partial z\partial w}\right|_{z_0,w_0} = 0. \tag{49}$$

Combining this with (46) gives $\left.\frac{\partial^2 P}{\partial z\partial w}\right|_{z_0,w_0} = 0$ contradicting (31). The next case is given by $1+r_1 = 3 \Rightarrow r_1 = 2$ and the condition for cancellation is

$$a_0 a_1 \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + a_1 \left.\frac{\partial^2 P}{\partial z\partial w}\right|_{z_0,w_0} + \frac{1}{6}\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} = 0 \tag{50}$$

from which

$$a_1 = \frac{1}{6}\left.\frac{\frac{\partial^3 P}{\partial z^3}}{\frac{\partial^2 P}{\partial z\partial w}}\right|_{z_0,w_0} \tag{51}$$

and (47) now becomes

$$\frac{1}{2}\sum_{\substack{i\geq 0\\ i+j\geq 2}}\sum_{j\geq 0}a_i a_j (z-z_0)^{r_i+r_j}\left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}+\sum_{k\geq 2}a_k(z-z_0)^{r_k+1}\left.\frac{\partial^2 P}{\partial z\partial w}\right|_{z_0,w_0}=0. \tag{52}$$

Now the most significant remaining terms are

$$\left(a_0 a_2 (z-z_0)^{1+r_2}+\frac{1}{2}a_1^2(z-z_0)^4\right)\left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}+a_2(z-z_0)^{1+r_2}\left.\frac{\partial^2 P}{\partial z\partial w}\right|_{z_0,w_0} \tag{53}$$

If $1+r_2 < 4$, the two terms with that power of $z-z_0$ must cancel giving

$$a_0 a_2 \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + a_2 \left.\frac{\partial^2 P}{\partial z\partial w}\right|_{z_0,w_0} = 0 \tag{54}$$

and using (46) and dividing by $a_2 \neq 0$ gives $\left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} = 0$ which is not possible. Clearly it is not possible for $4 < 1 + r_2$ in (53) because that would require $a_1 = 0$, and the only other possibility is $1 + r_2 = 4$ and the cancellation of all the terms in (53) simplifies to

$$a_2 = \frac{1}{72} \left.\frac{\left(\frac{\partial^3 P}{\partial z^3}\right)^2 \left(\frac{\partial^2 P}{\partial w^2}\right)}{\left(\frac{\partial^2 P}{\partial z \partial w}\right)^3}\right|_{z_0,w_0} \tag{55}$$

The next most significant terms now remaining are

$$\left[a_0 a_3 (z - z_0)^{1+r_3} + a_1 a_2 (z - z_0)^5\right] \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + a_3 (z - z_0)^{1+r_3} \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} \tag{56}$$

If $1 + r_3 < 5$, the condition that most significant terms are now cancelling is

$$a_0 a_3 \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + a_3 \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} = 0 \tag{57}$$

which when divided by $a_3$ and using (46) again contradicts (33). If $5 < 1 + r_3$ the cancellation of the leading term now gives $a_1 a_2 = 0$ which is not possible. Therefore $1 + r_3 = 5$ and the cancellation of all the leading terms simplifies to

$$a_3 = \frac{1}{2^4.3^3} \left.\frac{\left(\frac{\partial^3 P}{\partial z^3}\right)^3 \left(\frac{\partial^2 P}{\partial w^2}\right)^2}{\left(\frac{\partial^2 P}{\partial z \partial w}\right)^5}\right|_{z_0,w_0} \tag{58}$$

This can be continued by induction with the assumptions that

$$\frac{1}{2} \sum_{\substack{i \geq 0 \\ i+j \geq l}} \sum_{j \geq 0} a_i a_j (z - z_0)^{r_i + r_j} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + \sum_{k \geq l} a_k (z - z_0)^{r_k + 1} \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} = 0 \tag{59}$$

is what remains of (40) and

$$r_i = i + 1 \text{ for } 0 \leq i \leq l - 1 \tag{60}$$

and that for $1 \leq i \leq l - 1$

$$a_i = \frac{\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0}^{i} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}^{i-1}}{\left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0}^{2i-1}} \beta_i \tag{61}$$

where the numbers $\beta_i > 0$ for all $i \geq 1$. The first step of the induction argument is to note that the most significant terms from each sum in (59) are

$$\begin{aligned} & a_0 a_l (z-z_0)^{1+r_l} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} \\ & \frac{1}{2} a_i a_{l-i} (z-z_0)^{l+2} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} \quad \text{for } 1 \leq i \leq l-1 \\ & a_l (z-z_0)^{1+r_l} \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} \end{aligned} \tag{62}$$

because from (60) it follows that the powers of $z - z_0$ for $i + j = l$ in the first double sum of (59) are $1+r_l$ and $l+2$ (for $(i,j) = (1, l-1), (2, l-2), \ldots (l-1, 1)$ $r_i + r_j = i + j + 2 = l + 2$ and if $(i,j) = (0,l)$ or $(l,0)$ $r_i + r_j = 1 + r_l$) and all other indices in that term are greater than either one (or both) of these values. The smallest index is the smaller of $l + 2$ and $r_l + 1$. Suppose the smallest index is $r_l + 1 < l + 2$ then the terms with exponent $r_l + 1$ cancel out. This requires

$$a_0 a_l \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + a_l \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} = 0 \tag{63}$$

and because $a_l \neq 0$, this leads to (49) and to a contradiction. The next case is when the smallest index is $l + 2 < r_l + 1$ then the terms giving exponent $l + 2$ cancel out. This gives $\sum_{i=1}^{l-1} a_i a_{l-i} = 0$ which is not possible because, using (61),

$$\sum_{i=1}^{l-1} a_i a_{l-i} = \frac{\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0}^{l} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}^{l-2}}{\left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0}^{2l-2}} \sum_{i=1}^{l-1} \beta_i \beta_{l-i} \tag{64}$$

and the last sum is $> 0$. Next suppose $r_l + 1 = l + 2$ i.e. $r_l = l + 1$ and $a_l$ is determined by

$$a_0 a_l \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + a_l \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} + \frac{1}{2} \sum_{i=1}^{l-1} a_i a_{l-i} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} = 0. \tag{65}$$

which can be solved for $a_l$ giving

$$a_l = \frac{\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0}^{l} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}^{l-1}}{\left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0}^{2l-1}} \beta_l \tag{66}$$

where

$$\beta_l = \frac{1}{2}\sum_{i=1}^{l-1}\beta_i\beta_{l-i} > 0. \tag{67}$$

This completes the induction proof and determines all the values of $a_l$ in the asymptotic expansion of (20) at the point $(z_0, w_0)$ for the case $r_0 = 1$. This is an example of how the analysis of the leading order terms is extended to all orders.

# 2 The Euclidean Algorithm for polynomials

A single step in this general process is as follows. The polynomial $A = \sum_{i=0}^{n} z^i a_i$ is divided by the polynomial $B = \sum_{i=0}^{n-1} z^i b_i$. Just considering the leading terms gives the quotient $\frac{za_n}{b_{n-1}}$ which when multiplied by $B$ gives $\frac{a_n}{b_{n-1}}\sum_{i=0}^{n-1} z^{i+1}b_i$ which when subtracted from $A$ gives the first remainder $\sum_{i=0}^{n-1} z^i\left(a_i - \frac{a_n b_{i-1}}{b_{n-1}}\right)$ where $b_{-1} = 0$. Therefore the leading terms remaining give the final part of the quotient as $\frac{1}{b_{n-1}}\left(a_{n-1} - \frac{a_n b_{n-2}}{b_{n-1}}\right)$ so the complete quotient $Q = z\frac{a_n}{b_{b-1}} + \frac{a_{n-1}}{b_{n-1}} - \frac{a_n b_{n-2}}{b_{n-1}^2}$ and the final remainder is

$$R = \sum_{i=0}^{n-2} z^i\left[\frac{a_n}{b_{n-1}}\left(\frac{b_{n-2}b_i}{b_{n-1}} - b_{i-1}\right) - \frac{b_i a_{n-1}}{b_{n-1}} + a_i\right]. \tag{68}$$

This can be checked by verifying that $A = BQ + R$.

The complete process, the Euclidean Algorithm, consists of replacing the equations

$$\begin{cases} A = 0 \\ B = 0 \end{cases} \tag{69}$$

by the equivalent system

$$\begin{cases} B = 0 \\ R = 0 \end{cases} \tag{70}$$

and repeating this process with $A$ replaced by $B$ and $B$ replaced by $R$ until $R$ has degree 0 in which case the final $R$ is identically zero if and only if (69) is consistent and its solution is then obtained from the final $B = 0$.

There is a generalisation to the above which happens when the order of the divisor polynomial is not related to that of the dividand. Suppose $A = \sum_{i=0}^{n} z^i a_i$ and $B = \sum_{i=0}^{m} z^i b_i$ and $m \leq n$. It is required to simplify the system

$$\begin{cases} A = 0 \\ B = 0 \end{cases} \tag{71}$$

i.e. determine if there are any common solutions, and if so find the lowest order polynomial equation giving them all. Write the quotient as $Q = \sum_{j=0}^{n-m} z^j \alpha_j$ then the general relationship $A = BQ + R$ can be written as

$$\sum_{i=0}^{n} z^i a_i = \left(\sum_{i=0}^{m} z^i b_i\right)\left(\sum_{j=0}^{n-m} z^j \alpha_j\right) + \sum_{i=0}^{m-1} z^i c_i \tag{72}$$

where the remainder $R$ is the last sum in (72). This can be written as

$$\sum_{l=0}^{n} z^l \left(\sum_{(i,j):i+j=l} \alpha_j b_i\right) + \sum_{i=0}^{m-1} z^i c_i = \sum_{i=0}^{n} z^i a_i \tag{73}$$

The range of the inner sum using $j$ as the discrete variable is given by $j = l - i$ where

$$0 \le j \le n - m \tag{74}$$

and $0 \le i \le m$ which after eliminating $i$ is

$$l - m \le j \le l. \tag{75}$$

Combining (74) and (75) gives $\max(0, l-m) \le j \le \min(l, n-m)$. This leads to two dichotomies, $l < m$ and the comparison of $l$ with $n - m$ (it does not matter which case $l = n - m$ is included with), therefore the distinct ranges of values of $l$ of interest are

$$\{0 \le l < m, m \le l \le n-m, n-m < l \le n\} \tag{76}$$

when $m \le n - m$ and

$$\{0 \le l < n-m, n-m \le l < m, m \le l \le n\} \tag{77}$$

when $m > n - m$, so this gives 6 cases to be considered. In general, equating powers of $z$ gives

$$a_l = c_l + \sum_j \alpha_j b_{l-j}. \tag{78}$$

This is to be solved for $\alpha_j$ and $c_l$, given $a_i$ and $b_i$ where $c_l = 0$ if $l \ge m$, and the specific cases can follow when the limits on $j$ in the 6 cases have been written down. Consider first the case $m \le n - m$. Then

$$a_l = c_l + \sum_{j=0}^{l} \alpha_j b_{l-j} \text{ for } 0 \le l < m. \tag{79}$$

$$a_l = \sum_{j=l-m}^{l} \alpha_j b_{l-j} \text{ for } m \le l \le n-m. \tag{80}$$

$$a_l = \sum_{j=l-m}^{n-m} \alpha_j b_{l-j} \text{ for } n-m < l \leq n. \tag{81}$$

The order in which these equations are solved for each variable can now be stated. The key is to look for (i) the first variable to be solved for which is $\alpha_{n-m}$ from the $l = n$ case of (81) and (2) the new variable included as $l$ changes by 1, noting that each new variable then depends only on previous variables that have been found. This leads to the order

$$\alpha_{n-m}, \alpha_{n-m-1}, \dots \alpha_{n-2m+1}, \alpha_{n-2m}, \dots \alpha_0, c_{m-1}, \dots c_0 \tag{82}$$

for $l$ in decreasing order from $n$ to 0 in (79),(80), and (81), apart from the fact that (81) could be solved in any order for the $c_l$. For convenience these equations are listed in order, solved for the variable of interest, followed by combining cases (80) and (81) as follows:

$$\alpha_{l-m} = \frac{a_l - \sum_{j=l-m+1}^{\min(l,n-m)} \alpha_j b_{l-j}}{b_m} \text{ for } l = n, n-1, \dots m \tag{83}$$

$$c_l = a_l - \sum_{j=0}^{l} \alpha_j b_{l-j} \text{ for } 0 \leq l < m \tag{84}$$

Similarly for the case $m > n - m$

$$\alpha_{l-m} = \frac{a_l - \sum_{j=l-m+1}^{n-m} \alpha_j b_{l-j}}{b_m} \text{ for } l = n, n-1, \dots m \tag{85}$$

and

$$c_l = a_l - \sum_{j=0}^{\min(l,n-m)} \alpha_j b_{l-j} \text{ for } 0 \leq l < m \tag{86}$$

In each case the first value to be solved for is for $l = n$ and gives $\alpha_{n-m} = \frac{a_n}{b_m}$. It is easy to check that these cases combine to give simply

$$\alpha_{l-m} = \frac{a_l - \sum_{j=l-m+1}^{\min(l,n-m)} \alpha_j b_{l-j}}{b_m} \text{ for } l = n, n-1, \dots m \tag{87}$$

$$c_l = a_l - \sum_{j=0}^{\min(l,n-m)} \alpha_j b_{l-j} \text{ for } 0 \leq l < m \tag{88}$$

regardless of which is the larger of $m$ and $n - m$. This completes a single step of the Euclidean Algorithm described above.

# 3 Applying the Euclidean algorithm to bivariate polynomials

Suppose $A$ and $B$ above are polynomials in $w$ as well as $z$. This implies that the coefficients above are all polynomials in $w$. The goal is again to determine whether or not the system is consistent and to simplify the system as much as possible. Then the above argument will lead in a single cycle to $R$ given by (86) where (85) the $a's$ and $b's$ are polynomials in $w$. Thus the $\alpha's$ and the $c's$ are all of the form of one polynomial in $w$ divided by another. Because $R$ is equated to zero, the fraction can be cleared so that $R = 0$ can be expressed as another bivariate polynomial equated to 0, and importantly of degree $m-1$ in $z$. Therefore the cycle can be repeated as above until the degree of $R$ is 0 in $z$, which is a polynomial in $w$ only equated to 0. This is the consistency condition and determines the value(s) of $w$ that are possible solutions. Then for each such $w$, the other equation generated by the algorithm is a polynomial in $z$ only that determines the set of possible values of $z$ consistent with that value of $w$. The results of this algorithm are unique, but the elimination could have been done the other way round by eliminating $w$ first, or indeed the roles of $z$ and $w$ could be exchanged at any cycle of the algorithm, and must give equivalent results.

# 4 Applying the Euclidean Algorithm to the search for singular points

From (15) and (16) equated to zero, having $z - z_0$ in place of $z$ in (72),

$$a_i = \begin{cases} 0 \text{ if } i \notin \Sigma \\ \dfrac{v_q}{s_q!} \text{ if } i = s_q \text{ for some } q \end{cases}, \tag{89}$$

$$b_i = \begin{cases} 0 & \text{if } i+1 \notin \Sigma \\ \dfrac{v_q}{(s_q-1)!} & \text{if } i+1 = s_q \text{ for some } q \end{cases} \tag{90}$$

where

$$d_q = \left.\frac{\partial^{s_q+t_q} P}{\partial z^{s_q} \partial w^{t_q}}\right|_{z_0,w_0} \tag{91}$$

and

$$v_q = \frac{(w-w_0)^{t_q} d_q}{t_q!}. \tag{92}$$

Here $q$ is unique (if it exists) for a given value of $i$. Applying the first cycle of Euclidean Algorithm gives, with $m = n-1$, and $l = n = s_k$,

$$\alpha_1 = \frac{a_n}{b_{n-1}} = \frac{a_{s_k}}{b_{s_k-1}} = \frac{\frac{1}{s_k!}v_k}{\frac{1}{(s_k-1)!}v_k} = \frac{1}{s_k}. \tag{93}$$

Then with $l = m$

$$\alpha_0 = \frac{a_{n-1} - \sum_{j=1}^{\min(n-1,1)} \alpha_j b_{n-1-j}}{b_{n-1}} = \frac{a_{n-1} - \alpha_1 b_{n-2}}{b_{n-1}} \tag{94}$$

provided $n > 1$. The condition needed i.e. $b_{n-1} \neq 0$ is $n \in \Sigma$ because (90) is correct because $n = s_k$ so $q = k$. The conditions in $a_{n-1}$ and $b_{n-2}$ are equivalent, so the expressions in $\alpha_0$ can be combined to give

$$\alpha_0 = \left\{ \begin{array}{cc} 0 & \text{if } n-1 \notin \Sigma \\ \dfrac{v_q}{v_k}\dfrac{(s_k-1)!}{(s_q-1)!}\left(\dfrac{1}{s_q} - \dfrac{1}{s_k}\right) & \text{if } n-1 = s_q \end{array} \right\}. \tag{95}$$

Then substituting for $a_i$, $b_i$ and $\alpha_1$ gives

$$\begin{aligned} c_i = &\left\{ \begin{array}{ll} 0 & \text{if } i \notin \Sigma \\ v_{q_1}\left(\dfrac{1}{s_{q_1}!} - \dfrac{1}{s_k(s_{q_1}-1)!}\right) & \text{if } i = s_{q_1} \end{array} \right\} \\ &- \left\{ \begin{array}{ll} 0 & \text{if } i+1 \notin \Sigma \\ \alpha_0 \dfrac{v_{q_2}}{(s_{q_2}-1)!} & \text{if } i+1 = s_{q_2} \end{array} \right\} \text{ for } 0 \le i < m. \end{aligned} \tag{96}$$

If $n-1 \in \Sigma$ then $n-1 = s_q$ for some $q$. Then $q = k$ leads to a contradiction because $s_k = n$ so $q \neq k$ and $\alpha_0 \neq 0$. Conversly $\alpha_0 \neq 0$ implies $n-1 \in \Sigma$. For applying the EA to

If $\alpha_0 = 0$ the second expression in braces in (96) vanishes, then the condition that $c_i = 0$, which is the condition of termination of the Euclidean algorithm(EA), reduces to $\frac{1}{s_{q_1}!} = \frac{1}{s_k(s_{q_1}-1)!}$ and to $s_{q_1} = s_k$, so $q_1 = k$ and this can only work for $i = s_{q_1} = n$ which is excluded from (96) therefore the Euclidean algorithm cannot terminate under this condition. These results show that whether $\alpha_0$ is 0 or not, is the first question to ask in the analysis of an example of (15).

If for any $i$ satisfying $0 \le i < m$ we have $i \in \Sigma$ and $i+1 \notin \Sigma$ then from (96) $c_i = v_{q_1}\left(\frac{1}{s_{q_1}!} - \frac{1}{s_k(s_{q_1}-1)!}\right) \neq 0$ because the bracket is zero only if $q_1 = k$, and the latter is already ruled out because $s_{q_1} = i$ and $s_k = n = m+1$. Therefore if termination occurs after one cycle of the Euclidean algorithm then $\Sigma = \{0, 1, \ldots n\}$ applies to the data input to this algorithm. Also, the termination condition then reduces to $v_{q_1}(1-i/n) = \alpha_0 v_{q_2}$, where $i = s_{q_1} = q_1$ and $i+1 = s_{q_2} = q_2$, and to $v_i(1-i/n) = \alpha_0 v_{i+1}$ for $0 \le i < m$.

The overall result so far is that there are 3 cases with the following conditions applying to the input arguments to one cycle of the EA.

Case 1: EA terminates and $\alpha_0 \neq 0$, and $\Sigma = \{0, 1, 2 \ldots, n\}$ and $v_{i+1} = v_i(1-i/n)/\alpha_0$ for $0 \le i < m-1$.

Case 2: $\alpha_0 \neq 0$, $n-1 \in \Sigma$ and the EA does not terminate.

Case 3: $\alpha_0 = 0$ and the EA does not terminate, and $n-1 \notin \Sigma$.

For applying the EA to (15) and(16), this can also be done using $w - w_0$ as the variable instead of $z - z_0$.

Consider the EA applied to bivariate polynomials in $z$ and $w$. Does $T_A$ and $T_B$ uniquely determine $T_R$ for one cycle? Characterise this relationship. Does the EA done to completion give a unique result regardless of which variable is used at any stage to take the part of $z$ in the EA?

# 5 Relaxing the condition of the polynomial being minimal

Suppose now that there are $q$ singular points, and each has associated with it $S$ and $T$ as described above with those properties, and the values of $\left.\frac{\partial^{i+j}P}{\partial z^i \partial w^j}\right|_{z_r, w_r}$ for $1 \le r \le q$. The question now is what are terms to be included in the polynomial $P$. Introducing the sets $S_p$ and $T_p$ with the same properties as $S$ and $T$ above, let

$$P(z, w) = \sum_{(k,l) \in S_p \cup T_p} \sum a_{kl} z^k w^l = 0 \tag{97}$$

implicitly define the multivalued analytic function $w(z)$ to be constructed having these properties at this set of singular points and no others. The logic of the previous section then follows leading to

$$\left.\frac{\partial^{i+j}P}{\partial z^i \partial w^j}\right|_{z_r, w_r} = \sum_{\substack{(k,l) \in S_r \cup T_r \\ k \ge i,\ l \ge j}} \sum a_{kl} \left( \frac{k! l! z_r^{k-i} w_r^{l-j}}{(k-i)!(l-j)!} \right) \quad \begin{array}{l} \text{for all } (i,j) \in S_r \cup T_r \\ \text{for } 1 \le r \le q \end{array} . \tag{98}$$

This is a system of equations for the $a_{kl}$ that does not have the nice properties that occurred in the case of a single singular point, but it can be brought into this form by repeated elimination of variables, though not uniquely, by different choices of $S_p$.

The system of equations (98) can be represented on a grid according to the pair $(i, j)$ at which $n_{ij}$ is the number of such equations. Each of those equations involves only the variables $a_{kl}$ where $k \ge i$ and $l \ge j$. The point

$(i, j)$ also represents the term in the polynomial $P$ involving $a_{ij}$ which is yet to be constructed because $S_p$ and $T_p$ have not yet been determined. If $n_{ij} > 1$, one of those equations (call it $e$) can have $a_{ij}$ eliminated from it. Then using one of the equations for $(i+1, j)$, $e$ can have $a_{i+1\ j}$ eliminated from it, likewise for $a_{i+2\ j}$,$a_{i+3\ j}$ etc. There can be no gap in the sequence of such eliminating equations because all the sets $S_r$ all satisfy (4). The result of this is that $e$ is now an equation involving only $a_{ij}$ for $k \geq i$ and $l \geq j+1$, thus it can move up the grid by one place in the direction of increasing $j$. The same argument can of course be made with $i$ and $j$ interchanged.

One approach to the elimination procedure is as follows: make moves from $e$ having $i = 0$ (if necesary) in order to obtain $\{(0, j) : n_{0j} \neq 0\} = \{S_p : i = 0\}$. All these moves are incrementing $j$ by 1. If this impossible the polynomial with $S_p$ cannot be constructed because $e$ can never move down in $i$. This should be done with the minimum number of moves so that all the values of $j$ remain as small as possible to maximise the chance of success. Now make single moves for each $e$ at $(0, j)$ such that $n_{0j} > 1$ in the order of increasing $j$, then all the non-zero values of $n_{0j}$ are 1. The condition (4) in the grid will not be altered by these moves. If for any resulting point $(i, j)$ for $e$ there is no corresponding term in $P$, it must be added to avoid the equations being overdetermined and there being no solution. Now do the same with $i$ and $j$ reversed. Now the whole procedure can be repeated for the column $j \geq i = 1$ then for the row $i \geq j = 1$ etc. in order to obtain the system such that $\{(1, j) : n_{1j} \neq 0\} = \{S_p : i = 1\}$ and $\{(i, 1) : n_{i1} \neq 0\} = \{S_p : j = 1\}$ etc.

The result of this is the original system (98) expressed in the form (12) or a proof of its impossibility.

By repeating these moves starting from (98) in all possible ways and keeping track of the numbers of equations at each grid point at each step until the resulting grid has no numbers $n_{ij} > 1$, a set of possible values of $S_p = \{(i, j) : n_{ij} = 1\}$ can be obtained, each with its corresponding value of $T_p$.

# 6 Extensions

In either of equations (**??**) or (**??**) if the functions $g_1()$ and $g_2()$ are not be single-valued (such as linear or bilinear functions) they could expressed like $f()$ in terms of single-valued functions. This suggests a recursive approach.

This would generate a set of types of behaviour at single singular points. In general for an analytic function there would be many such singular points, and the behaviours thus described would be approximate or asymptotic being modified by the effect of the other singular points. This is in analogy with the behaviour of algebraic functions. Also it would be very desirable to be able to extract the above types of asymptotic behaviours from analytic functions defined indirectly eg as integrals or solutions of differential or integral equations. This could probably be done in analogy with $\Delta w = a\Delta z^r$ for algebraic functions by replacing this with other relationships for which $g_1()$ and $g_2()$ can be found and $\Delta z = 0 \Rightarrow \Delta w = 0$ e.g. $\Delta w = a(\Delta z)^{r_1}(\ln \Delta z)^{r_2}$ Or the general problem: Given $f()$, directly or indirectly, with a singular point at $z_0$ say, find the functions $g_1()$ and $g_2()$ satisfying (**??**) or (**??**) or other functions defining them, for $z$ close to $z_0$. Note that (**??**) can have $z$ replaced by $z_0$ to generate an equation of the form (**??**) when analysing in the neighbourhood of $z_0$.

# 7 More general classes of analytic functions

Because of the elimination theorem, any algebraic function can be written with the use of redundant variables in the following form

$$P_i(z, w, x_1, \ldots x_{n-1}) = 0 \text{ for } 0 \leq i \leq n \tag{99}$$

where the $P_i$ are multivariate polynomials and the (complex) variables $x_i$ are to be eliminated from the system resulting in a single equation of the form $P(z, w) = 0$. In few examples that I have studied, actually carrying out the stated elimination is extremely complicated and as such it may frequently be more convenient to manipulate the function in the form (99) rather than attempt the actual elimination to the form $P(z, w) = 0$ let alone the explicit algebraic formula (if it exists), using implicit function methods.

Furthermore this form suggests the extension to functions $w(z)$ defined by the following elimination problem where the $P_i$ are polynomial functions of all their arguments:

$$P_i(z, w, x_1, \ldots x_{n-1}, e^{x_1}, \ldots e^{x_{n-1}}) = 0 \text{ for } 0 \leq i \leq n \tag{100}$$

may be an interesting extension of algebraic functions, regardless of whether or not such an elimination can xbe done explicitly. A simple example of this is the $n$th iterate of the exponential function which can be written in this form as

$$\begin{array}{l} x_1 - \exp(z) = 0 \\ x_2 - \exp(x_1) = 0 \\ \ldots \\ x_{n-1} - \exp(x_{n-2}) = 0 \\ w - \exp(x_{n-1}) = 0 \end{array} \tag{101}$$

but not in this form for a smaller value of $n$ showing that as the *depth* $n$ of the system increases, more functions are included in the form (100). The depth could be defined as zero when $w$ is expressed explicitly in terms of $z$ by a formula.

# 8 Deriving the conditions for singular points in terms of derivatives of the $P_i$

Returning to a simpler case, suppose a analytic function $w(z)$ is expressed not merely implicitly by

$$P(z,w) = 0 \tag{102}$$

but even more implicitly by

$$\begin{cases} P_1(z,w,x) = 0 \\ P_2(z,w,x) = 0 \end{cases} \tag{103}$$

from which $x$ is to be eliminated. The question is if the analytic function is defined by the form (103) how can these defining equations for singular points be expressed? One way to approach this is to write the general equations (to first order) relating the infinitesimal changes in the variables in the two different ways of expressing this relationship, and eliminate $\Delta x$ from the system arising from (**??**) and compare it with the relationship between $\Delta z$ and $\Delta w$ only, arising from (**??**). This gives

$$\frac{\partial P}{\partial z}\Delta z + \frac{\partial P}{\partial w}\Delta w = 0 \tag{104}$$

and

$$\begin{aligned} \frac{\partial P_1}{\partial z}\Delta z + \frac{\partial P_1}{\partial w}\Delta w + \frac{\partial P_1}{\partial x}\Delta x = 0 \\ \frac{\partial P_2}{\partial z}\Delta z + \frac{\partial P_2}{\partial w}\Delta w + \frac{\partial P_2}{\partial x}\Delta x = 0 \end{aligned} \tag{105}$$

from which elimination of $\Delta x$ gives

$$\Delta z\left(\frac{\partial P_2}{\partial z} - \frac{\partial P_1}{\partial z}\frac{\frac{\partial P_2}{\partial x}}{\frac{\partial P_1}{\partial x}}\right) + \Delta w\left(\frac{\partial P_2}{\partial w} - \frac{\partial P_1}{\partial w}\frac{\frac{\partial P_2}{\partial x}}{\frac{\partial P_1}{\partial x}}\right) = 0 \tag{106}$$

and comparing (104) with (106) gives

$$\frac{\partial P}{\partial z}\bigg/\left|\frac{\partial(P_1,P_2)}{\partial(x,z)}\right| = \frac{\partial P}{\partial w}\bigg/\left|\frac{\partial(P_1,P_2)}{\partial(x,w)}\right| \tag{107}$$

where the denominators are determinants of the Jacobian matrices of partial derivatives, and (1) and (2) can be represented by

$$\left|\frac{\partial(P_1,P_2)}{\partial(x,z)}\right| = 0 \tag{108}$$

and

$$\left|\frac{\partial(P_1,P_2)}{\partial(x,w)}\right| = 0 \tag{109}$$

respectively. Note that neither of these Jacobian determinants can go to infinity because the $P_i$ and their derivatives, being polynomials, are all finite at finite values of $z$ and $w$, hence finite $x$. Extending this argument to higher derivative conditions for singular points proved to be a little tricky.

Adding in the second order terms in the relationships amongst the infinitesimal changes to the variables, which are the leading terms omitted from (104) in the Taylor expansion of $P$, gives

$$\frac{\partial P}{\partial z}\Delta z + \frac{\partial P}{\partial w}\Delta w + \frac{\partial^2 P}{\partial z^2}\frac{\Delta z^2}{2} + \frac{\partial^2 P}{\partial z \partial w}\Delta z \Delta w + \frac{\partial^2 P}{\partial w^2}\frac{\Delta w^2}{2} = 0. \tag{110}$$

Likewise for (105) in the Taylor expansion of $P_1$ and $P_2$:

$$\begin{array}{l} \dfrac{\partial P_i}{\partial z}\Delta z + \dfrac{\partial P_i}{\partial w}\Delta w + \dfrac{\partial P_i}{\partial x}\Delta x + \dfrac{\partial^2 P_i}{\partial z^2}\dfrac{\Delta z^2}{2} + \dfrac{\partial^2 P_i}{\partial z \partial w}\Delta z \Delta w + \dfrac{\partial^2 P_i}{\partial z \partial x}\Delta z \Delta x + \\ \dfrac{\partial^2 P_i}{\partial w \partial x}\Delta w \Delta x + \dfrac{\partial^2 P_i}{\partial w^2}\dfrac{\Delta w^2}{2} + \dfrac{\partial^2 P_i}{\partial x^2}\dfrac{\Delta x^2}{2} = 0 \text{ for } i \in \{1, 2\} \end{array}. \tag{111}$$

In this pair of quadratic equations for $\Delta x$, consistency requires that the linear combination of these that is linear in $\Delta x$ is also satisfied. This can be written as

$$\Delta x = -\left(\frac{1}{2}\Delta z^2 a + \Delta z \Delta w B + \frac{1}{2}\Delta w^2 C + F\Delta z + G\Delta w\right) \bigg/ (\Delta z D + \Delta w E + H) \tag{112}$$

where

$$A = \begin{vmatrix} \dfrac{\partial^2 P_1}{\partial x^2} & \dfrac{\partial^2 P_2}{\partial x^2} \\ \dfrac{\partial^2 P_1}{\partial z^2} & \dfrac{\partial^2 P_2}{\partial z^2} \end{vmatrix} \qquad B = \begin{vmatrix} \dfrac{\partial^2 P_1}{\partial x^2} & \dfrac{\partial^2 P_2}{\partial x^2} \\ \dfrac{\partial^2 P_1}{\partial z \partial w} & \dfrac{\partial^2 P_2}{\partial z \partial w} \end{vmatrix} \qquad C = \begin{vmatrix} \dfrac{\partial^2 P_1}{\partial x^2} & \dfrac{\partial^2 P_2}{\partial x^2} \\ \dfrac{\partial^2 P_1}{\partial w^2} & \dfrac{\partial^2 P_2}{\partial w^2} \end{vmatrix}$$
$$D = \begin{vmatrix} \dfrac{\partial^2 P_1}{\partial x^2} & \dfrac{\partial^2 P_2}{\partial x^2} \\ \dfrac{\partial^2 P_1}{\partial z \partial x} & \dfrac{\partial^2 P_2}{\partial z \partial x} \end{vmatrix} \qquad E = \begin{vmatrix} \dfrac{\partial^2 P_1}{\partial x^2} & \dfrac{\partial^2 P_2}{\partial x^2} \\ \dfrac{\partial^2 P_1}{\partial w \partial x} & \dfrac{\partial^2 P_2}{\partial w \partial x} \end{vmatrix} \qquad F = \begin{vmatrix} \dfrac{\partial P_1}{\partial z} & \dfrac{\partial P_2}{\partial z} \\ \dfrac{\partial^2 P_1}{\partial x^2} & \dfrac{\partial^2 P_2}{\partial x^2} \end{vmatrix}$$
$$G = \begin{vmatrix} \dfrac{\partial P_1}{\partial w} & \dfrac{\partial P_2}{\partial w} \\ \dfrac{\partial^2 P_1}{\partial x^2} & \dfrac{\partial^2 P_2}{\partial x^2} \end{vmatrix} \qquad H = \begin{vmatrix} \dfrac{\partial P_1}{\partial x} & \dfrac{\partial P_2}{\partial x} \\ \dfrac{\partial^2 P_1}{\partial x^2} & \dfrac{\partial^2 P_2}{\partial x^2} \end{vmatrix} \tag{113}$$

Comparing (112) with (111) from which it was derived, it is clear that (112) can be cancelled down to an expression linear in the differentials otherwise back substitution would lead to expressions involving 4th powers of $\Delta z$. It is straightforward to identify the result as

$$\Delta x = -\frac{A}{2D}\Delta z - \frac{C}{2E}\Delta w \tag{114}$$

using the highest power terms in the numerator of (112). Substituting this back into say the first of (111) (the second would give the an equivalent result because the the consistency between them, (112), has already been taken into account) gives a result of the form (110) and comparing the coefficients of the differentials in these equations shows that the following relations have to be satisfied:

$$\begin{aligned}
\frac{\partial^2 P}{\partial z^2} &\propto \frac{\partial^2 P_1}{\partial z^2} + \frac{A^2}{4D^2}\frac{\partial^2 P_1}{\partial x^2} - \frac{A}{2D}\frac{\partial^2 P_1}{\partial z \partial x} \\
\frac{\partial^2 P}{\partial w^2} &\propto \frac{\partial^2 P_1}{\partial w^2} + \frac{C^2}{4E^2}\frac{\partial^2 P_1}{\partial x^2} - \frac{C}{2E}\frac{\partial^2 P_1}{\partial w \partial x} \\
\frac{\partial^2 P}{\partial z \partial w} &\propto \frac{\partial^2 P_1}{\partial z \partial w} + \frac{AC}{4DE}\frac{\partial^2 P_1}{\partial x^2} - \frac{A}{2D}\frac{\partial^2 P_1}{\partial w \partial x} - \frac{C}{2E}\frac{\partial^2 P_1}{\partial z \partial x} \\
\frac{\partial P}{\partial z} &\propto -\frac{A}{2D}\frac{\partial P_1}{\partial x} \\
\frac{\partial P}{\partial w} &\propto -\frac{C}{2E}\frac{\partial P_1}{\partial x}
\end{aligned} \tag{115}$$

where the constant of proportionality is the same for each case.

These equations are very complicated, and even more so when higher order terms are considered, so it it might be better when dealing with examples to do the eliminations to obtain $\Delta x$ and (110) to obtain the coefficients which are the derivatives of $P$ rather than using the general formulae. The suggested procedure is this: first write down the derivatives of $P_i$ to the order needed. Do the elimination between the system (111) to obtain $\Delta x$. Substitute this back into one of (111) to obtain (110) and read off the derivatives of $P$ needed.

How many derivatives of $P$ are needed w.r.t. $w$ and $z$? The point is to obtain all the singular points so the search must start as follows:

- Find all the points where (1) $\partial P/\partial z = 0$.
- Find all the points where (2) $\partial P/\partial w = 0$. Then for the second order derivatives:
- For each answer to (1), (1.1) find all points where also $\partial^2 P/\partial z^2 = 0$.
- For each answer to (2), (2.1) find all points where also $\partial^2 P/\partial w^2 = 0$.
- For each common answer to (1) and (2), (2.2) find all any points where also $\partial^2 P/\partial z \partial w = 0$. Then for 3rd order derivatives:

- For each answer to (1.1), find all points where also $\partial^3 P/\partial z^3 = 0$.
- For each common answer to (1.1) and (2.2) find all points where also $\partial^3 P/\partial z^2 \partial w = 0$.
- For each common answer to (2.1) and (2.2) find all points where also $\partial^3 P/\partial z \partial w^2 = 0$.
- For each answer to (2.1), find all points where also $\partial^3 P/\partial w^3 = 0$. etc..

This could be continued indefinitely and ensures that the condition attached to Equation (3) holds. The result of this search is the list of all the singular points. The leading order non-zero derivatives for each such point must also be found. The values of $a$ and $r$ in the leading order expression $\Delta w = a \Delta z^r$ can then be obtained [Nixon2013] for each singular point.

Given all the pairs of values of $a$ and $r$ for a singular point at $(z_0, w_0)$ can the leading order non-zero derivatives of $P$ at $(z_0, w_0)$ be obtained?

## References

[Churchill et al.] Churchill R.V., Brown J.W., Verhey R.F., Complex Variables and Applications, Third Edition, McGraw-Hill Kogakusha 1974

[Nixon2013] Nixon J., Theory of algebraic functions on the Riemann Sphere Mathematica Aeterna Vol. 3, 2013, no. 2, 83-101 https://www.longdom.org/articles/theory-of-algebraic-functions-on-the-riemann-sphere.pdf