This document is a work in progress. As such it is incomplete and still has errors and omissions. When brought to a state where I cannot easily find any improvements it will form my next paper on Complex analysis.

The section ending at *********** at the top of page 19 has now been checked.

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Towards a Theory of Analytic Functions

Abstract

Prior thesis: Multivalued analytic functions defined on the Riemann Sphere, and not constrained by added boundaries or constraints on values, are determined uniquely by their behaviours at all their singular points. This is now believed to be not quite correct because the way that they could interact is not included. The range and nature of such interactions will be investigated but is believed to result from how the multiple values in the neighbourhoods of the singular points are joined up. I will also consider types of behaviour at singular points that go beyond the behaviours associated with singular points of algebraic functions. The emphasis here will be how to handle multi-valued functions in calculations rather than the topological properties of the surface representing such functions.

Mathematics Subject Classification:

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My earlier work on algebraic functions indicates that their topology when considered as multivalued functions on the Riemann Sphere $\mathbb{C} \cup \{\infty\} = \mathbb{C}^*$ determines them uniquely apart from a "strength" constant associated with each branch point or pole (formerly denoted by A but here I use a). This set of functions includes the identity function $z \to z$ and constant functions $z \to c$ for any $c \in \mathbb{C}$ and was closed under the following unary and binary operations on functions:

1. union
2. addition
3. multiplication
4. composition
5. inversion

6. differentiation

This has the unfortunate consequence that $c \to z$ for the single value $c \in \mathbb{C}^*$ for all $z \in \mathbb{C}^*$ and $z \to \emptyset$ for all $z \neq c$, the inverse of the constant function, $z \to c$ is included, but this will probably not be a problem, i.e. I am sure that this has a resolution. However it seems probable that the inclusion of this and other "pathological" functions in the algebra accounts for the lack of general acceptance of this way of approaching complex analysis.

This naturally raises the question of the extension of these results to include indefinite integration as an operation that gives closure. This requires functions ln and exp to be included and some functions with singular points that are not poles or branch points known as essential singular points.

1 A more general description of branch points for algebraic functions

Consider $f(z) = z^{1/q}$ where q is an integer. Rather than describing this behaviour simply by saying that it is expressed by a "winding number", near the branch point at $z = 0$, is to relate $f(z)$ to f evaluated at the "next" branch of the function obtained by tracking $f(z)$ continuously round a small circle round $z = 0$ described in the anticlockwise direction until the same point is reached. This will have to be described q times to get back to the same value of $f(z)$. So let $g_1(z) = e^{2\pi i/q}z$. Then it is easy to show that $f(z) = g_1(f(z))$ and if z goes round the origin q times, $f(z)$ will go round the origin once to come back to the same value. In fact this equation represents the equality of the two sets of values each being q in number. This relationship is a better way of describing this situation because it just involves the single-valued function g_1 and no mention of topological concepts that are not so easy to make precise. It is natural to consider the question whether this uniquely determines f apart from a constant factor. Definitely $f(z)^q$ has no branch point at $z = 0$.

Now consider $f(z) = z^p$ where p is an integer. Then similarly $f(z) = f(g_2(z))$ where $g_2(z) = e^{2\pi i/p}z$ and if z goes round the origin, $f(z)$ will go round the origin p times to come back to the same value. This relationship just involves the single-valued function g_2.

Next consider $f(z) = z^{p/q}$ where p and q are integers. In general this will have to be described q times to get back to the same value of $f(z)$. By combining the previous results it is obvious to try $g_2(z) = e^{2\pi i/p}z$ and $g_1(z) = e^{-2\pi i/q}z$. Then it is easy to show that $f(z) = g_1(f(g_2(z)))$ and if z goes round the origin q times, $f(z)$ will go round the origin p times to come back to the same value. This relationship just involves the single-valued functions $g_1()$ and $g_2()$. Because $f()$ is multivalued this implies that the sets of values of $f()$ on the left and right are related as indicated without exception.

Theorem 1.1. *Let $f(z)$ satisfy*

$$f(z) = g_1(f(g_2(z))). \tag{1}$$

Let $k()$ and $l()$ be single valued functions with inverses that are also single-valued then the same relationship holds with $f()$ replaced by $k(f(l()))$, $g_1()$ replaced by $k(g_1(k^{-1}()))$ and $g_2()$ replaced by $l^{-1}(g_2(l()))$.

Proof. Making these substitutions gives the same relationship with the function $k()$ applied to both sides and expressed in terms of the independent variable $w = l^{-1}(z)$. □

Note that any relationship of the form (1) etc. does not specify the location of the singularity. It just relates the function values to themselves after an arbitrary path for z back to itself that could pass round a singularity resulting possibly in finishing on a different sheet of the multi-sheeted surface that corresponds to $f()$. For example suppose $k(z) = az + b$ and $l(z) = cz + d$ then the function $f^*(z) = af(cz + d) + b$ satisfies $f^*(z) = g_1^*(f^*(g_2^*(z)))$ i.e.(1) with $g_1^*(z) = ag_1((z - b)/a) + b$ and $g_2^*(z) = (g_2(cz + d) - d)/c$ relating this analysis for an analytic with singularity at $z = 0$ to a similar analysis for a function with a singular point other than at 0.

If (1) holds then it also holds with $f(z)$ replaced by If in equation (1) $g_1()$ and $g_2()$ are also single valued and $g_1^{-1}()$ is applied to both sides and the result expressed in terms of the variable $w = g_2(z)$ then the same relationship holds with $g_1()$ replaced by $g_1^{-1}()$ and $g_2()$ replaced by $g_2^{-1}()$.

The inverse functions of both sides of Equation (1) again give an equation of the same form but with $g_1()$ replaced by g_2^{-1} and $g_2()$ replaced by $g_1^{-1}()$.

Let $z = l(w)$ where w is a new complex variable, and express (1) by the equality of $k()$ applied to both sides, and express it in terms of the variable w

$k(f(z)) = k(g_1(f(g_2(z))))$. This can be written as $f^*(z) = g_1^*(f^*(g_2(z)))$ where $f^*(z) = k(f(z))$ and $g_1^*(z) = k(g_1(k^{-1}(z)))$. Also introducing the new variable w by $z = l(w)$ where $l()$ is also a function of the same type as $k()$ then from (1) $f(l(w)) = g_1(f(g_2(l(w))))$ i.e. $f^+(w) = g_1(f^+(g_2^+(w)))$ where $f^+(w) = f(l(w))$ and $g_2^+(w) = l^{-1}(g_2(l(w)))$.

2 A survey of examples

Analysis of behaviour in the neighbourhood of singular points (there is only one finite singular point in these examples) similar to the above can be found for functions of a complex variable that are not algebraic as the following examples show.

Probably the simplest example is $f(z) = w = \ln(z)$ the inverse of the complex exponential function. Because this is equivalent to $z = \exp(w) =$

$\exp(w).\exp(2\pi i) = \exp(w + 2\pi i)$. So $w + 2\pi i = \ln(z)$ and equation (1) is satisfied for $f() = \ln()$ and $g_2(z) = z + 2\pi i$ and $g_1(z) = z$. As in the examples above $g_1()$ and $g_2()$ are single-valued and the singular point of $f()$ is at $z = 0$.

Consider $w = (\ln(z))^2$. Can a similar analysis for this be done? We have $w = (\ln(z) + 2\pi i)^2$ so if in Theorem 1.1 $k(z) = z^2$ and $l(z) = z$ then it works formally with $g_1(z) = (z^{1/2} + 2\pi i)^2$ and $g_2(z) = z$. Note that $g_1()$ is now not single-valued. Another analysis of this sort comes from $\ln(z)^2 = (-\ln(z))^2 = (\ln(z^{-1}))^2$ i.e. Equation (1) with $g_1(z) = z$ and $g_2(z) = z^{-1}$, which shows that if in Equation (1) either of $g_1()$ or $g_2()$ is not single-valued, this analysis may not be unique.

Consider $f(z) = z\ln(z)$. Then $f(z) = z\ln(z) + 2\pi iz$. This can be represented in terms similar to (1) with single valued g_1 and g_2 but this time the equation takes the slightly more general form

$$f(z) = g_1(z, f(g_2(z))) \tag{2}$$

in which g_1 has direct z dependance in addition to its dependence on $f()$. In this case $g_1(z, f) = 2\pi iz + f$ and $g_2(z) = z$.

Suppose $f(z) = a(\ln(z))^3$. Put $g_2(z) = z^{2\pi i/3} = \exp(\frac{2\pi i}{3}\ln(z))$ then $f(g_2(z)) = a(\frac{2\pi i}{3}\ln z)^3 = a(\ln z)^3(\frac{2\pi i}{3})^3$ and if $g_1(z, f) = f(\frac{2\pi i}{3})^{-3}$ then (2) holds. In this case $g_2(z)$ is not single-valued. An obvious approach might be to now try to define $g_2()$ by another equation of the form (2), so this $g_2()$ will now be called $h()$. Then $h(z) = \exp(\frac{2\pi i}{3}(\ln z + 2\pi i)) = \exp(-4\pi^2/3).\exp(\frac{2\pi i}{3}\ln z) = \exp(-4\pi^2/3)h(z)$, that is $h(z)$ is satisfies (2) with $g_1^*(z, f) = \exp(-4\pi^2/3).f$ and $g_2^*(z) = z$. This finally gives an analysis for $f()$ involving two instances of (2) and single-valued functions $g_1()$, $g_1^*()$ and $g_2^*()$.

Consider solutions of

$$f(z) = f(z)^{1/2}. \tag{3}$$

This is of course equivalent to $f(z) = f(z)^2$, and is equation (1) with $g_1(f) = f^2$ and $g_2(z) = z$. In this case $g_1(f)$ is single-valued but its inverse is not. Introducing $k(z) = \ln(\ln f(z))2\pi i/\ln(2)$ so $k()$ must satisfy $k(z) = k(z) + 2\pi i$ and so this is satisfied by $k(z) = a + \ln(z + b)$ where a and b are arbitrary constants. Inverting this gives

$$f(z) = \exp\left(c\exp\left(\frac{\ln(2)}{2\pi i}.\ln(z+b)\right)\right). \tag{4}$$

where c is an arbitrary constant related to a. Notice that (3) does not require the singular point to be at $z = 0$, but at $f = 0$, and the singular point of $f()$ is at $z = -b$.

In this last example too the defining equations for the singular point can be expressed in terms of single-valued functions because $g_1(f) = f^2$ (and actually $g_1(f) = af^2 + b$ for arbitrary constants a and b) satisfies (1) with $g_1^*(z) = z$,$g_2^*(z) = -z$.

3 Uniqueness of the solutions to equations (1) (2)

The above analysis would be far more useful if it was known whether or not the equation obtained in the form of (1) or (2) has a unique solution for $f()$. (Note that this is not to be confused with a function being multivalued so for example it is possible that the two-valued function $z^{1/2}$ is the only function that satisfies a condition, then it is said to be unique although 2-valued.) A very simple example shows that this is not generally so. Suppose $f(z) = f(-z)$ then this is satisfied by $f_1(z) = z^2$ and by $f_2(z) = z^4$ and in fact any function of z^2. A simple trick may be useful here. Suppose the condition (in general of the form (2)) is required to be an inequality unless equality is explicitly required, then in the above example $f(z_1) \neq f(z_2)$ unless $z_1 = -z_2$ i.e.

$$\begin{array}{l} z_1 = -z_2 \Rightarrow f(z_1) = f(z_2) \\ z_1 \neq -z_2 \Rightarrow f(z_1) \neq f(z_2) \end{array} . \tag{5}$$

Now the question is does (5) have the unique solution set $f(z) = az^2 + b$ for arbitrary constants a and b or what extra conditions are needed for it to be so? I will add the conditions that

- $f()$ has no singular points except possibly at 0 and ∞, and
- $f(0) = 0$

Consider the function $k(z) = f(z^{1/2})$. Suppose e.g. $z^{1/2} \in$ UHP (the upper half plane given by $0 \leq \theta \leq \pi$ where θ is the angular coordinate in the complex plane). Then by (99) all values of $k(z)$ for different z are different and $k()$ has no singular points except possibly at 0 and ∞ and $k(0) = 0$. This would seem to imply that $k()$ also has no singular point at $z = 0$ and at ∞ and hence is a linear function because singular points are associated with multiple related function values as given by (1) or (2) but it doesn't seem easy to prove it. However $k(z) = az + b$ is a solution implying $f(z^{1/2}) = az + b$ i.e. $f(w) = aw^2 + b$ and $b = 0$ so $f(w) = aw^2$. That there could be other solutions will be left open for now and I return to the description of types of complex singular points.

4 Functions with a single singular point

The simplest conditions for a singular point at (z_0, w_0) are either

$$\frac{\partial P}{\partial z}(z_0, w_0) = 0 \tag{6}$$

or

$$\frac{\partial P}{\partial w}(z_0, w_0) = 0. \tag{7}$$

These conditions can be extended by adding to them

$$\left.\frac{\partial^{s+t} P}{\partial z^s \partial w^t}\right|_{z_0, w_0} = 0 \tag{8}$$

for any $(s,t) \in S$ where the set S satisfies

$$\begin{array}{l}(s,t) \in S \text{ implies both } (s-1,t) \in S \text{ (provided } s-1 \geq 0) \\ \text{and } (s,t-1) \in S \text{ (provided } t-1 \geq 0).\end{array} \tag{9}$$

These conditions ensure that a derivative is not equated to zero when a more significant derivative (corresponding to a more significant i.e. lower order term in the Taylor series) at the same point is non-zero. There is another set of derivatives that dominate (are of lower order than) any other derivative not required to be zero. These are the derivatives in (8) for all $(s,t) \in T$ where T is such that

$$\begin{array}{l}\text{for each } (s,t) \text{ such that } s \geq 0 \text{ and } t \geq 0 \text{ and } (s,t) \notin S \text{ then} \\ \text{for at least one } (k,l) \in T, s \geq k \text{ and } t \geq l.\end{array} \tag{10}$$

Then S is uniquely determined by T as the set (s,t) such that for all $(k,l) \in T, s < k$ or $t < l$ i.e.

$$S = \{(s,t) : \forall (k,l) \in T (0 \leq s < k \text{ or } 0 \leq t < l)\} \tag{11}$$

The set T is also unique once S is determined. This is because each member of T imposes a condition on S and none of these conditions can be deduced from the others (if that happened the deduced ones would be removed from T), so if $(s_1, t_1) \in T$ this implies that $0 \leq s < s_1$ or $0 \leq t < t_1$ so this condition must therefore be in any alternative T_1 to T that has the same effect (same S) or deduced from it, but the latter is ruled out because any T is defined to be minimal as described above. This shows that any member of T is in T_1 and vice versa therefore $T = T_1$ so T is unique.

For finding the derivatives of the polynomial the following result is needed:

$$\frac{\partial^k z^s}{\partial z^k} = \frac{s! z^{s-k}}{(s-k)!} \text{ if } k \leq s \text{ and } 0 \text{ otherwise} \tag{12}$$

so

$$\frac{\partial^{k+l}}{\partial z^k \partial w^l}(z^s w^t) = \begin{cases} \dfrac{s!t!z^{s-k}w^{t-l}}{(s-k)!(t-l)!} & \text{if } k \leq s \text{ and } l \leq t \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

Now consider what terms need to be included in the polynomial $P(z,w) = \sum\sum a_{st}z^s w^t = 0$ that represents the function $w(z)$. If $(s_0,t_0) \in T$ and $a_{s_0t_0} = 0$ then the required non-zero value for $\frac{\partial^{s_0+t_0}P}{\partial z^{s_0}\partial w^{t_0}}\Big|_{z_0,w_0}$ can only come from term(s) $a_{st}z^s w^t$ where $s \geq s_0$ and $t \geq t_0$. Therefore the simplest (lowest order) choice of polynomial is when $a_{st} \neq 0$ for all $(s,t) \in T$ and $a_{st} = 0$ for all non-negative integer pairs $(s,t) \notin S \cup T$. This gives

$$P(z,w) = \sum_{(k,l)\in S\cup T}\sum a_{kl}z^k w^l = 0. \tag{14}$$

There are presumably interesting cases with polynomials not satisfying (14) when more than one singular point is expected, but the following analysis concerns only cases when (14) holds. The following system

$$\left.\frac{\partial^{s+t}P}{\partial z^s\partial w^t}\right|_{z_0,w_0} = 0 \text{ for all } (s,t) \in S \tag{15}$$

involving the parameters

$$\left.\frac{\partial^{s+t}P}{\partial z^s\partial w^t}\right|_{z_0,w_0} \quad \text{for } (s,t) \in T \tag{16}$$

must be solved for the a_{st} for $(s,t) \in S \cup T$. Sustituting (14) into $\frac{\partial^{s+t}P}{\partial z^s\partial w^t}$ and using (13) gives

$$\left.\frac{\partial^{s+t}P}{\partial z^s\partial w^t}\right|_{z_0,w_0} = \sum_{\substack{(k,l)\in S\cup T\\ k\geq s,\ l\geq t}}\sum a_{kl}\left(\frac{k!l!z_0^{k-s}w_0^{l-t}}{(k-s)!(l-t)!}\right) \text{ for all } (s,t) \in S \cup T. \tag{17}$$

For $(s,t) \in T$ there is just a single term in the sum. It has $k = s$ and $l = t$ so

$$a_{st} = \frac{1}{s!t!}\left.\frac{\partial^{s+t}P}{\partial z^s\partial w^t}\right|_{z_0,w_0} \quad \text{for all } (s,t) \in T. \tag{18}$$

Equation (17) relates a_{st} to only other values a_{kl} with $k \geq s$ and $l \geq t$. If the latter have already been found, a_{st} can be determined. The latter themselves can be solved from other members of (17) likewise. Therefore if for each $(s,t) \in S \cup T$

$$n = \#\{(k,l) \in S \cup T : k \geq s \text{ and } l \geq t\} \tag{19}$$

is introduced, every element a_{st} can be solved for in terms of other a_{kl} with a smaller value of n. Therefore (17) must be solved for the a_{st} in any order in which n is non-decreasing. This shows that the a_{st} are uniquely determined from (17).

The result of this is

$$P(z,w) = \sum_{(s,t)\in T} \frac{(z-z_0)^s}{s!}\frac{(w-w_0)^t}{t!}\left.\frac{\partial^{s+t}P}{\partial z^s \partial w^t}\right|_{z_0,w_0} \tag{20}$$

because this is the Taylor series expansion of P, using (15), about (z_0, w_0) truncated so that no terms with $(z-z_0)^k(w-w_0)^l$ such that $k > s$ or $l > t$ for any $(s,t) \in T$ contribute in accordance with (14). To consider singular points the following derivative is also needed

$$\frac{\partial P}{\partial z} = \sum_{(s,t)\in T,\ s>0} \frac{(z-z_0)^{s-1}}{(s-1)!}\frac{(w-w_0)^t}{t!}\left.\frac{\partial^{s+t}P}{\partial z^s \partial w^t}\right|_{z_0,w_0}. \tag{21}$$

The following notation will be used for $(s,t) \in T$ where $\#(T) = k+1$,

$$T = \{(s_0,t_0),(s_1,t_1),\dots(s_k,t_k)\} \tag{22}$$

where the $s's$ increase with the subscript and $t's$ decrease with the subscript i.e. $q < r$ implies $s_q < s_r$ and $t_q > t_r$ and $s_0 = 0$ and $t_k = 0$. It is also convenient to indroduce $\Sigma = \{s_0, s_1 \dots s_k\}$. To answer the question of whether (20) has any singular points other than (z_0, w_0), the Euclidean algorithm will be used with (20) and (21), regarding these as polynomials in $z - z_0$. At the first step (20) is divided by (21) just considering the leading powers of $z - z_0$. The first quotient and remainder are obtained removing an overall factor $w - w_0$, then (21) takes the place of (20) and the the remainder takes the place of (21) and this is repeated until 0 is obtained. The previous remainder is the necessary and sufficient condition under which both (20) and (21) hold i.e. one of the conditions for a singular point other than when $w = w_0$.

For example $T = \{(0,2),(1,1),(3,0)\}$. This implies (20) takes the form

$$P(z,w) = \frac{(w-w_0)^2}{2}\left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + (z-z_0)(w-w_0)\left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} + \frac{(z-z_0)^3}{6}\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} \tag{23}$$

from which follows

$$\frac{\partial P}{\partial z} = (w-w_0)\left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} + \frac{(z-z_0)^2}{2}\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0}. \tag{24}$$

Both the (23) and (24) individually equated to 0 show that if $w = w_0$ then $z = z_0$ and conversly because the partial derivatives in (20) and (23) are all non-zero. Eliminating the cubic term in $z - z_0$ shows that because of (24) equated to 0, (23) can be replaced by (25)

$$P - \frac{(z-z_0)}{3}\frac{\partial P}{\partial z} = \frac{(w-w_0)^2}{2}\left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + \frac{2}{3}(z-z_0)(w-w_0)\left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} \tag{25}$$

When trying to find common solutions to (24) and (25) equated to 0 with $z \neq z_0$ (and therefore $w \neq w_0$), $w - w_0$ can be factored out giving

$$\frac{(w-w_0)}{2}\left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + \frac{2}{3}(z-z_0)\left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} = 0 \tag{26}$$

so common solutions to (26) and (24) must be found. Eliminating the highest powers of $z - z_0$ and again taking out the $w - w_0$ factor gives

$$z - z_0 = \frac{8}{3}\frac{\left.\frac{\partial^2 P}{\partial z \partial w}^2\right|_{z_0,w_0}}{\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0}\left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}} \tag{27}$$

This combined with (26) gives

$$w - w_0 = -\frac{32}{9}\frac{\left.\frac{\partial^2 P}{\partial z \partial w}^3\right|_{z_0,w_0}}{\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0}\left.\frac{\partial^2 P}{\partial w^2}^2\right|_{z_0,w_0}} \tag{28}$$

Checking this solution ((27) and (28) which will be denoted by (z_1, w_1)) by substituting it back shows that (23) and (24) are satisfied and the assumption that $w \neq w_0$ is eliminating the other solution $z = z_0, w = w_0$. In a similar way the common solution of $P = \frac{\partial P}{\partial w} = 0$ other than $z = z_0, w = w_0$ was obtained by treating $w - w_0$ as the variable giving

$$z - z_0 = 3\frac{\left.\frac{\partial^2 P}{\partial z \partial w}^2\right|_{z_0,w_0}}{\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0}\left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}} \tag{29}$$

$$w - w_0 = -3\frac{\left.\frac{\partial^2 P}{\partial z \partial w}^3\right|_{z_0,w_0}}{\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0}\left.\frac{\partial^2 P}{\partial w^2}^2\right|_{z_0,w_0}} \tag{30}$$

which will be denoted by (z_2, w_2). The similarity of these results is surprising and gives

$$\begin{array}{l} z_1 - z_0 = \frac{2^3}{3^2}(z_2 - z_0) \\ w_1 - w_0 = \frac{2^5}{3^3}(w_2 - w_0) \end{array} . \tag{31}$$

The single singular point expected is obtained if $\left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} = 0$.

In order to determine the sets S (9) and T and the leading terms in the expansion of w in terms of z for each singular point, derivatives will be needed evaluated at (z_1, w_1) and (z_2, w_2) as well as at (z_0, w_0). Starting with the main singular point (z_0, w_0), an expansion of the form

$$w - w_0 = a_1(z - z_0)^{r_1} + a_2(z - z_0)^{r_2} \dots \tag{32}$$

will be sought. The terms will be in decreasing order of significance i.e. $r_1 < r_2 \ldots$.

For (z_1, w_1) from (24), the terms cancel giving

$$\left.\frac{\partial P}{\partial z}\right|_{z_1,w_1} = 0 \tag{33}$$

and

$$\frac{\partial^2 P}{\partial z \partial w} = \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} \neq 0 \tag{34}$$

and

$$\frac{\partial^2 P}{\partial z^2} = (z - z_0) \left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} \neq 0, \tag{35}$$

and from (23)

$$\frac{\partial P}{\partial w} = (w - w_0) \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + (z - z_0) \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} \tag{36}$$

which at (z_1, w_1) becomes

$$\frac{8}{9} \frac{\left.\frac{\partial^2 P}{\partial z \partial w}^3\right|_{z_0,w_0}}{\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}} \neq 0. \tag{37}$$

Therefore for (z_1, w_1), $T = \{(0,1), (2,0)\}$ and $S = \{(0,0), (1,0)\}$. and the leading term in the expansion of $w - w_1$ is

$$w - w_1 = -\frac{9}{16} \frac{\left.\frac{\partial^3 P}{\partial z^3} \frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}}{\left.\frac{\partial^2 P}{\partial z \partial w}^2\right|_{z_0,w_0}} (z - z_1)^2. \tag{38}$$

For (z_2, w_2), (36) implies

$$\frac{\partial^2 P}{\partial w^2} = \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} \neq 0, \tag{39}$$

and (34), and (24) at (z_2, w_2) becomes

$$\frac{3}{2} \frac{\left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0}}{\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} \left.\frac{\partial^2 P}{\partial w^2}^2\right|_{z_0,w_0}} \neq 0 \tag{40}$$

so $S = \{(0,0), (0,1)\}$ so no extra derivatives for either singular point are zero.

Equation (23) is quadratic for w and so can be solved for w in terms of z. Write (23) as $A(w - w_0)^2 + B(w - w_0) + C = 0$ where $A = \frac{1}{2} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}$, $B = (z - z_0) \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0}$ and $C = \frac{(z-z_0)^3}{6} \left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0}$. Writing (23) as a quadratic for w alone shows that the descriminant is still $B^2 - 4AC$ and the solution is

$$w = w_0 + \frac{(z_0 - z)\left[\left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} + \left(\left.\frac{\partial^2 P}{\partial z \partial w}^2\right|_{z_0,w_0} - \frac{1}{3}(z - z_0)\left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}\left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0}\right)^{1/2}\right]}{\left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}}. \tag{41}$$

To check this, (27) and one of (41) is in agreement with (28), and (29) and (41) implies (30).

Next a series expansion of the form

$$w - w_0 = \sum_{i \geq 1} a_i (z - z_0)^{r_i} \tag{42}$$

will be developed for the singular point at (z_0, w_0) where $a_i \neq 0$ and the terms are ordered so that $i < j$ implies $r_i < r_j$. These conditions insure that the terms are in decreasing order of significance for values of z close to z_0. First substitute (42) into (23) giving

$$\begin{aligned} &\frac{1}{2}\sum_{i\geq 1}\sum_{j \geq 1} a_i a_j (z - z_0)^{r_i + r_j} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + \sum_{k \geq 1} a_k (z - z_0)^{r_k + 1} \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} \\ &+ \frac{(z - z_0)^3}{6} \left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} = 0 \end{aligned} . \tag{43}$$

Therefore for each power of $z - z_0$ appearing in (43), the coefficients must total to zero. Although the r_i have not yet been determined, this is possible with the following strategy. The most significant i.e. lowest order terms in each of the 3 expressions resulting from the 3 terms in (23) must each be cancelled by terms (not necessarily the lowest order) from another of those expressions. Working back from any term that cancels another to the most significant term in the set and which cancels it etc. the first terms to be considered are the most significant terms the whole of (43) which must cancel in pairs (or threes etc.). To do this, the powers of $z - z_0$ in the leading terms from the sets of terms derived from (23) are tested for equality in every pairwise combination and cancellation requires 2 or more of these to be equal and smaller than any of the other powers of $z - z_0$ in the set of other leading terms. This could happen in more than one way. If this condition can be satisfied, then there is an equation to be satisfied ensuring cancellation of these terms occurs. Then this repeated with the remaining terms of (43) etc. to determine all the coefficients a_i and r_i.

Carrying this out for (43) shows that the leading terms have powers of $z - z_0$ equal to $2r_1$,$r_1 + 1$ and 3 and cancellation requires at least two of these to be equal, so equating all possible combinations gives

- $2r_1 = r_1 + 1 \Rightarrow r_1 = 1$, and both sides are equal to 2 and the other leading term has 3, so the leading terms can be cancelled.
- $2r_1 = 3 \Rightarrow r_1 = 3/2$. Both sides of the equation are 3 and the other leading term has index $r_1 + 1 = 5/2$ which is more significant so this more significant term could not be cancelled subsequently therefore this value of r_1 cannot be used.
- $r_1 + 1 = 3 \Rightarrow r_1 = 2$, and the other leading exponent is $2r_1 = 4$ which is less significant than these and could be cancelled subsequently.

For $r_1 = 1$, the cancellation of the leading terms requires

$$\frac{1}{2}a_1^2 \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + a_1 \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} = 0 \tag{44}$$

from which the non-zero solution is

$$a_1 = \frac{-2 \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0}}{\left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}} \tag{45}$$

$$\begin{aligned}&\frac{1}{2}\sum_{\substack{i\geq 1 \\ i+j\geq 3}}\sum_{j\geq 1} a_i a_j (z-z_0)^{r_i+r_j} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + \sum_{k\geq 2} a_k (z-z_0)^{r_k+1} \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} \\ &+\frac{(z-z_0)^3}{6} \left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} = 0\end{aligned} . \tag{46}$$

The most significant terms are now

$$\left\{ a_1 a_2 (z-z_0)^{1+r_2} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0}, a_2 (z-z_0)^{1+r_2} \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0}, \frac{(z-z_0)^3}{6} \left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} \right\} \tag{47}$$

The next possible cancellation is for index $1+r_2$ and requires $a_1 a_2 \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + a_2 \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} = 0$ which leads either to $a_2 = 0$ which is excluded or

$$a_1 \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} = 0. \tag{48}$$

Combining this with (45) gives $\left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} = 0$ contradicting (34). The next case is given by $1 + r_2 = 3 \Rightarrow r_2 = 2$ and the condition for cancellation is

$$a_1 a_2 \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + a_2 \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} + \frac{1}{6} \left.\frac{\partial^3 P}{\partial z^3}\right|_{z_0,w_0} = 0 \tag{49}$$

from which

$$a_2 = \frac{1}{6} \left.\frac{\frac{\partial^3 P}{\partial z^3}}{\frac{\partial^2 P}{\partial z \partial w}}\right|_{z_0,w_0} \tag{50}$$

and (46) now becomes

$$\frac{1}{2} \sum_{\substack{i \geq 1 \\ i+j \geq 4}} \sum_{j \geq 1} a_i a_j (z - z_0)^{r_i + r_j} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + \sum_{k \geq 3} a_k (z - z_0)^{r_k + 1} \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} = 0. \tag{51}$$

This can be continued by induction with the assumption that

$$\frac{1}{2} \sum_{\substack{i \geq 1 \\ i+j \geq l+1}} \sum_{j \geq 1} a_i a_j (z - z_0)^{r_i + r_j} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + \sum_{k \geq l} a_k (z - z_0)^{r_k + 1} \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} = 0 \tag{52}$$

is what remains of (43) and $r_i = i$ for $1 \leq i \leq l - 1$. It follows that the powers of $z - z_0$ for $i + j = l + 1$ in the first double sum of (52) are $1 + r_l$ and $l + 1$ (for $(i, j) = (2, l - 1), (3, l - 2), \ldots (l - 1, 2)$ $r_i + r_j = i + j = l + 1$ and if $(i, j) = (1, l)$ or $(l, 1)$ $r_i + r_j = 1 + r_l$) and all other indices in that term are greater than either one (or both) of these values and the smallest index is the smaller of $l + 1$ and $r_l + 1$. Suppose the smallest index is $r_l + 1 < l + 1$ then the terms with exponent $r_l + 1$ cancel out. This leads to

$$a_1 a_l \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + a_l \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} = 0 \tag{53}$$

and because $a_l \neq 0$, to (48) and to a contradiction. Next suppose the smallest index is $l + 1 < r_l + 1$ then the terms giving exponent $l + 1$ cancel out. This gives $\frac{1}{2} \sum_{i=2}^{l-1} a_i a_{l+1-i} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} = 0$ which is impossible because all the a's and derivatives of P in (23) are non-zero. Therefore it follows that $r_l + 1 = l + 1$ i.e. $r_l = l$ and a_l is determined by

$$a_1 a_l \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} + a_l \left.\frac{\partial^2 P}{\partial z \partial w}\right|_{z_0,w_0} + \frac{1}{2} \sum_{i=2}^{l-1} a_i a_{l+1-i} \left.\frac{\partial^2 P}{\partial w^2}\right|_{z_0,w_0} = 0. \tag{54}$$

This completes the induction proof that $r_l = l$ for all positive integers l and determines all the values of a_l. This is an example of how the analysis of the leading order terms is extended to all orders.

****

A single step in this general process is as follows. The polynomial $A = \sum_{i=0}^{n} z^i a_i$ is divided by the polynomial $B = \sum_{i=0}^{n-1} z^i b_i$. Just considering the leading terms gives the quotient $\frac{z a_n}{b_{n-1}}$ which when multiplied by B gives $\frac{a_n}{b_{n-1}} \sum_{i=0}^{n-1} z^{i+1} b_i$ which when subtracted from A gives the first remainder $\sum_{i=0}^{n-1} z^i \left(a_i - \frac{a_n b_{i-1}}{b_{n-1}} \right)$ where $b_{-1} = 0$. Therefore the leading terms remaining give the final part of the quotient as $\frac{1}{b_{n-1}} \left(a_{n-1} - \frac{a_n b_{n-2}}{b_{n-1}} \right)$ so the complete quotient $Q = z \frac{a_n}{b_{b-1}} + \frac{a_{n-1}}{b_{n-1}} - \frac{a_n b_{n-2}}{b_{n-1}^2}$ and the final remainder is

$$R = \sum_{i=0}^{n-2} z^i \left[\frac{a_n}{b_{n-1}} \left(\frac{b_{n-2} b_i}{b_{n-1}} - b_{i-1} \right) - \frac{b_i a_{n-1}}{b_{n-1}} + a_i \right]. \tag{55}$$

This can be checked by verifying that $A = BQ + R$.

The complete process, the Euclidean Algorithm, consists of replacing the equations

$$\begin{cases} A = 0 \\ B = 0 \end{cases} \tag{56}$$

by the equivalent system

$$\begin{cases} B = 0 \\ R = 0 \end{cases} \tag{57}$$

and repeating this process with A replaced by B and B replaced by R until R has degree 0 in which case the final R is identically zero if and only if (56) is consistent and its solution is then obtained from the final $B = 0$.

A little work shows that using this algorithm to develop a formula for the result becomes very rapidly unmanageable as n increases. It is much easier to formulate this by working backwards from the simplest case when $n = 2$. In this case

$$R = a_0 - \frac{b_0}{b_1} \left(a_1 - a_2 \frac{b_0}{b_1} \right). \tag{58}$$

This works if $b_1 \neq 0$, but if $b_1 = 0$ a solution satisfying $B = 0$ is only possible if in addition $b_0 = 0$ in which case the system reduces to $A = 0$. For the case $n = 3$, if $b_2 \neq 0$ the first iteration gives the remainder

$$R = z \left[a_1 - \frac{b_0}{b_2} a_3 - \frac{b_1}{b_2} \left(a_2 - \frac{b_1}{b_2} a_3 \right) \right] + a_0 - \frac{b_0}{b_2} \left(a_2 - \frac{b_1}{b_2} a_3 \right). \tag{59}$$

The second and final step for solving the system is to repeat the calculation with $A = z^2 b_2 + z b_1 + b_0$ and $B = R$ as in (59). This gives (58) with the following substitutions:

$$\begin{aligned} &a_0 \to b_0 \\ &a_1 \to b_1 \\ &a_2 \to b_2 \\ &b_0 \to a_0 - \tfrac{b_0}{b_2}\left(a_2 - \tfrac{b_1}{b_2}a_3\right) \\ &b_1 \to a_1 - \tfrac{b_0}{b_2}a_3 - \tfrac{b_1}{b_2}\left(a_2 - \tfrac{b_1}{b_2}a_3\right) \end{aligned} \tag{60}$$

where only one substitution is performed for each symbol e.g. the b_0 in the substitution for a_0 in line 1 is not further substituted for b_0 in line 4.

Suppose that the original problem has been solved to obtain the final R for polynomials A and B of degrees $n-1$ and $n-2$ respectively. Then to solve this problem for polynomials A and B of degrees n and $n-1$ respectively, the first remainder is obtained as above giving R as in (55) then applying the general method to B and R in place of A and B as supposed above. This is done by substituting in the final R, $b_i \to$ the coefficient of z^i in (55) and $a_i \to b_i$. Putting this together shows that the general problem of finding the final R given $A = \sum_{i=0}^{n} z^i a_i$ and $B = \sum_{i=0}^{n-1} z^i b_i$ is solved by starting with (58) and doing $n-2$ substitutions with the j'th substitution being

$$\begin{aligned} &a_i \to b_i \text{ for } 0 \le i \le j+1 \text{ and} \\ &b_i \to \frac{a_n}{b_{n-1}}\left(\frac{b_{n-2}b_i}{b_{n-1}} - b_{i-1}\right) - \frac{b_i a_{n-1}}{b_{n-1}} + a_i \text{ for } 0 \le i \le j \end{aligned} \tag{61}$$

such that each symbol is substituted only once. Here the $j=1$ case is of course the same as (60). Note that numerical evaluation must be done in the reverse order of the substitutions.

There is a generalisation to the above which happens when the order of the divisor polynomial is not related to that of the dividand. Suppose $A = \sum_{i=0}^{n} z^i a_i$ and $B = \sum_{i=0}^{m} z^i b_i$ and $m \le n$. It is required to simplify the system

$$\begin{cases} A = 0 \\ B = 0 \end{cases} \tag{62}$$

i.e. determine if there are any common solutions, and if so find the lowest order polynomial equation giving them all. Write the quotient as $Q = \sum_{j=0}^{n-m} z^j \alpha_j$ then the general relationship $A = BQ + R$ can be written as

$$\sum_{i=0}^{n} z^i a_i = \left(\sum_{i=0}^{m} z^i b_i\right)\left(\sum_{j=0}^{n-m} z^j \alpha_j\right) + \sum_{i=0}^{m-1} z^i c_i \tag{63}$$

where the remainder R is the last sum in (63). This can be written as

$$\sum_{l=0}^{n} z^l \left(\sum_{(i,j):i+j=l} \alpha_j b_i\right) + \sum_{i=0}^{m-1} z^i c_i = \sum_{i=0}^{n} z^i a_i \tag{64}$$

The range of the inner sum using j as the discrete variable is given by $j = l - i$ where

$$0 \leq j \leq n - m \tag{65}$$

and $0 \leq i \leq m$ which after eliminating i is

$$l - m \leq j \leq l. \tag{66}$$

Combining (65) and (66) gives $\max(0, l - m) \leq j \leq \min(l, n - m)$. This leads to two dichotomies, $l < m$ and the comparison of l with $n - m$ (it does not matter which case $l = n - m$ is included with), therefore the distinct ranges of values of l of interest are

$$\{0 \leq l < m, m \leq l \leq n - m, n - m < l \leq n\} \tag{67}$$

when $m \leq n - m$ and

$$\{0 \leq l < n - m, n - m \leq l < m, m \leq l \leq n\} \tag{68}$$

when $m > n - m$, so this gives 6 cases to be considered. In general, equating powers of z gives

$$a_l = c_l + \sum_j \alpha_j b_{l-j}. \tag{69}$$

This is to be solved for α_j and c_l, given a_i and b_i where $c_l = 0$ if $l \geq m$, and the specific cases can follow when the limits on j in the 6 cases have been written down. Consider first the case $m \leq n - m$. Then

$$a_l = c_l + \sum_{j=0}^{l} \alpha_j b_{l-j} \text{ for } 0 \leq l < m. \tag{70}$$

$$a_l = \sum_{j=l-m}^{l} \alpha_j b_{l-j} \text{ for } m \leq l \leq n - m. \tag{71}$$

$$a_l = \sum_{j=l-m}^{n-m} \alpha_j b_{l-j} \text{ for } n - m < l \leq n. \tag{72}$$

The order in which these equations are solved for each variable can now be stated. The key is to look for (i) the first variable to be solved for which is α_{n-m} from the $l = n$ case of (72) and (2) the new variable included as l changes by 1, noting that each new variable then depends only on previous variables that have been found. This leads to the order

$$\alpha_{n-m}, \alpha_{n-m-1}, \ldots \alpha_{n-2m+1}, \alpha_{n-2m}, \ldots \alpha_0, c_{m-1}, \ldots c_0 \tag{73}$$

for l in decreasing order from n to 0 in (70),(71), and (72), apart from the fact that (72) could be solved in any order for the c_l. For convenience these equations are listed in order, solved for the variable of interest, followed by combining cases (71) and (72) as follows:

$$\alpha_{l-m} = \frac{a_l - \sum_{j=l-m+1}^{\min(l,n-m)} \alpha_j b_{l-j}}{b_m} \text{ for } l = n, n-1, \ldots m \tag{74}$$

$$c_l = a_l - \sum_{j=0}^{l} \alpha_j b_{l-j} \text{ for } 0 \leq l < m \tag{75}$$

Similarly for the case $m > n - m$

$$\alpha_{l-m} = \frac{a_l - \sum_{j=l-m+1}^{n-m} \alpha_j b_{l-j}}{b_m} \text{ for } l = n, n-1, \ldots m \tag{76}$$

and

$$c_l = a_l - \sum_{j=0}^{\min(l,n-m)} \alpha_j b_{l-j} \text{ for } 0 \leq l < m \tag{77}$$

In each case the first value to be solved for is for $l = n$ and gives $\alpha_{n-m} = \frac{a_n}{b_m}$. It is easy to check that these cases combine to give simply

$$\alpha_{l-m} = \frac{a_l - \sum_{j=l-m+1}^{\min(l,n-m)} \alpha_j b_{l-j}}{b_m} \text{ for } l = n, n-1, \ldots m \tag{78}$$

$$c_l = a_l - \sum_{j=0}^{\min(l,n-m)} \alpha_j b_{l-j} \text{ for } 0 \leq l < m \tag{79}$$

regardless of which is the larger of m and $n - m$. This completes a single step of the Euclidean Algorithm described above.

5 Applying the Euclidean Algorithm to the search for singular points

From (20) and (21) equated to zero,

$$a_i = \begin{cases} 0 \text{ if } i \notin \Sigma \\ \dfrac{v_q}{s_q!} \text{ if } i = s_q \text{ for some } q \end{cases} \tag{80}$$

$$b_i = \begin{cases} 0 & \text{if } i+1 \notin \Sigma \\ \dfrac{v_q}{(s_q-1)!} & \text{if } i+1 = s_q \text{ for some } q \end{cases} \tag{81}$$

where

$$d_q = \left.\frac{\partial^{s_q+t_q} P}{\partial z^{s_q} \partial w^{t_q}}\right|_{z_0,w_0} \tag{82}$$

and

$$v_q = \frac{(w-w_0)^{t_q} d_q}{t_q!} \tag{83}$$

Applying the Euclidean Algorithm first gives, for $m = n-1$, and $l = n = s_k$,

$$\alpha_1 = \frac{a_n}{b_{n-1}} = \frac{a_{s_k}}{b_{s_k-1}} = \frac{\frac{1}{s_k!} d_k}{\frac{1}{(s_k-1)!} d_k} = \frac{1}{s_k} \tag{84}$$

Then

$$\alpha_0 = \frac{a_{n-1} - \sum_{j=1}^{\min(n-1,1)} \alpha_j b_{n-1-j}}{b_{n-1}} = \frac{a_{n-1} - \alpha_1 b_{n-2}}{b_{n-1}} \tag{85}$$

provided $n > 1$. The condition that $b_{n-1} \neq 0$ is $n \in \Sigma$ which is correct because $n = s_k$ so $q = k$. The conditions in a_{n-1} and b_{n-2} are equivalent, so the expressions in α_0 can be combined to give

$$\alpha_0 = \left\{ \begin{array}{cc} 0 & \text{if } n-1 \notin \Sigma \\ \dfrac{v_q}{v_k} \dfrac{(s_k-1)!}{(s_q-1)!} \left(\dfrac{1}{s_q} - \dfrac{1}{s_k} \right) & \text{if } n-1 = s_q \end{array} \right\}. \tag{86}$$

Then

$$c_i = \left\{ \begin{array}{cc} 0 & \text{if } i \notin \Sigma \\ \dfrac{v_q}{s_q!} & \text{if } i = s_q \end{array} \right\} - \left\{ \begin{array}{cc} 0 & \text{if } i = 0 \\ \dfrac{b_{i-1}}{s_k} & \text{if } i > 0 \end{array} \right\} - b_i \alpha_0 \tag{87}$$

Then evaluating this expression for all 6 compatible combinations of the conditions on i involved in (87) gives

Conditions on i			c_i
$i = 0$		$i+1 \notin \Sigma$	v_0
$i = 0$		$i+1 = s_q$	$v_0 + \frac{v_q}{(s_q-1)!}$
$i > 0$	$i \notin \Sigma$	$i+1 \notin \Sigma$	0
$i > 0$	$i \notin \Sigma$	$i+1 = s_q$	$-\alpha_0 \frac{v_q}{(s_q-1)!}$
$i > 0$	$i = s_q$	$i+1 \notin \Sigma$	$v_q(\frac{1}{s_q!} - \frac{1}{s_k(s_q-1)!})$
$i > 0$	$i = s_{q_1}$	$i+1 = s_{q_2}$	$v_{q_1} \left(\frac{1}{s_{q_1}!} - \frac{1}{s_k(s_{q_1}-1)!} \right) - \alpha_0 \frac{v_{q_2}}{(s_{q_2}-1)!}$

(88)

This completes the first cycle of the Eucldean Algorithm for searching for singular points in the general case. ***************** The first quotient is

$$\frac{\frac{(z-z_0)^{i_k}}{i_k!}\left.\frac{\partial^{i_k}P}{\partial z^{i_k}}\right|_{z_0,w_0}}{\frac{(z-z_0)^{i_k-1}}{(i_k-1)!}\left.\frac{\partial^{i_k}P}{\partial z^{i_k}}\right|_{z_0,w_0}} = \frac{(z-z_0)}{i_k} \tag{89}$$

and the remainder simplifies to

$$\frac{\sum_{(i,j)\in T, i>0, j>0}(z-z_0)^i\frac{(w-w_0)^{j-1}}{j!}\left.\frac{\partial^{i+j}P}{\partial z^i\partial w^j}\right|_{z_0,w_0}\left(\frac{1}{i!}-\frac{1}{i_k(i-1)!}\right)+}{\frac{(w-w_0)^{j_k-1}}{j_k!}\left.\frac{\partial^{j_k}P}{\partial w^{j_k}}\right|_{z_0,w_0}} = 0 \tag{90}$$

The next quotient is

$$\frac{\frac{(z-z_0)^{i_k-1}}{(i_k-1)!}\left.\frac{\partial^{i_k}P}{\partial z^{i_k}}\right|_{z_0,w_0}}{(z-z_0)^{i_{k-1}}\left.\frac{\partial^{i_{k-1}+1}P}{\partial z^{i_{k-1}}\partial w}\right|_{z_0,w_0}\left(\frac{1}{i_{k-1}}-\frac{1}{i_k(i_{k-1}-1)!}\right)} \tag{91}$$

When dividing the polynomial $P_1 = \sum_{i=0}^{n} a_i z^i$ by $P_2 = \sum_{i=0}^{n-1} b_i z^i$ the procedure is like long division so the first quotient from the leading terms is $(a_n/b_{n-1})z$ then multiplying this by P_2 and subtracting from P_1 gives the remainder from which the constant term in the quotient is obtained giving $(a_n/b_{n-1})(1-b_{n-2}/b_{n-1})$. The final remainder is

The other possibility is obtained with w and z interchanged when $P = \frac{\partial P}{\partial w} = 0$ is being solved. Can (14) be relaxed without giving rise to any more singular points?

Also if there was another singular point generated by P at (z_1, w_1) say, then some other set of equations of the form (17) would hold with z_0 and w_0 replaced by z_1 and w_1 with the same range of indices (k, l) but with a possibly different set of points (i, j) corresponding to the type of that singular point. Considering the same procedure for solving the system in the same order, no equation in that system can be distinct from the corresponding one in the system for (z_0, w_0) without an inconsistency arising. Therefore the singular point (z_0, w_0) is unique.

6 Multiple singular points

Suppose now that there are q singular points, and each has associated with it S and T as described above with those properties, and the values of $\left.\frac{\partial^{i+j}P}{\partial z^i\partial w^j}\right|_{z_r,w_r}$ for $1 \leq r \leq q$. The question now is what are terms to be included in the polynomial

P. Introducing the sets S_p and T_p with the same properties as S and T above, let

$$P(z,w) = \sum\sum_{(k,l)\in S_p\cup T_p} a_{kl} z^k w^l = 0 \tag{92}$$

implicitly define the multivalued analytic function $w(z)$ to be constructed having these properties at this set of singular points and no others. The logic of the previous section then follows leading to

$$\left.\frac{\partial^{i+j}P}{\partial z^i \partial w^j}\right|_{z_r,w_r} = \sum\sum_{\substack{(k,l)\in S_r\cup T_r\\ k\geq i,\ l\geq j}} a_{kl}\left(\frac{k!l!z_r^{k-i}w_r^{l-j}}{(k-i)!(l-j)!}\right) \begin{array}{l}\text{for all } (i,j)\in S_r\cup T_r\\ \text{for } 1\leq r\leq q\end{array}. \tag{93}$$

This is a system of equations for the a_{kl} that does not have the nice properties that occurred in the case of a single singular point, but it can be brought into this form by repeated elimination of variables, though not uniquely, by different choices of S_p.

The system of equations (93) can be represented on a grid according to the pair (i,j) at which n_{ij} is the number of such equations. Each of those equations involves only the variables a_{kl} where $k\geq i$ and $l\geq j$. The point (i,j) also represents the term in the polynomial P involving a_{ij} which is yet to be constructed because S_p and T_p have not yet been determined. If $n_{ij}>1$, one of those equations (call it e) can have a_{ij} eliminated from it. Then using one of the equations for $(i+1,j)$, e can have $a_{i+1\ j}$ eliminated from it, likewise for $a_{i+2\ j}$,$a_{i+3\ j}$ etc. There can be no gap in the sequence of such eliminating equations because all the sets S_r all satisfy (9). The result of this is that e is now an equation involving only a_{ij} for $k\geq i$ and $l\geq j+1$, thus it can move up the grid by one place in the direction of increasing j. The same argument can of course be made with i and j interchanged.

One approach to the elimination procedure is as follows: make moves from e having $i=0$ (if necesary) in order to obtain $\{(0,j):n_{0j}\neq 0\}=\{S_p:i=0\}$. All these moves are incrementing j by 1. If this impossible the polynomial with S_p cannot be constructed because e can never move down in i. This should be done with the minimum number of moves so that all the values of j remain as small as possible to maximise the chance of success. Now make single moves for each e at $(0,j)$ such that $n_{0j}>1$ in the order of increasing j, then all the non-zero values of n_{0j} are 1. The condition (9) in the grid will not be altered by these moves. If for any resulting point (i,j) for e there is no corresponding term in P, it must be added to avoid the equations being overdetermined and there being no solution. Now do the same with i and j reversed. Now the whole procedure can be repeated for the column $j\geq i=1$ then for the row $i\geq j=1$ etc. in order to obtain the system such that $\{(1,j):n_{1j}\neq 0\}=\{S_p:i=1\}$ and $\{(i,1):n_{i1}\neq 0\}=\{S_p:j=1\}$ etc.

The result of this is the original system (93) expressed in the form (17) or a proof of its impossibility.

By repeating these moves starting from (93) in all possible ways and keeping track of the numbers of equations at each grid point at each step until the resulting grid has no numbers $n_{ij} > 1$, a set of possible values of $S_p = \{(i,j) : n_{ij} = 1\}$ can be obtained, each with its corresponding value of T_p.

7 Extensions

In either of equations (1) or (2) if the functions $g_1()$ and $g_2()$ are not be single-valued (such as linear or bilinear functions) they could expressed like $f()$ in terms of single-valued functions. This suggests a recursive approach.

This would generate a set of types of behaviour at single singular points. In general for an analytic function there would be many such singular points, and the behaviours thus described would be approximate or asymptotic being modified by the effect of the other singular points. This is in analogy with the behaviour of algebraic functions. Also it would be very desirable to be able to extract the above types of asymptotic behaviours from analytic functions defined indirectly eg as integrals or solutions of differential or integral equations. This could probably be done in analogy with $\Delta w = a\Delta z^r$ for algebraic functions by replacing this with other relationships for which $g_1()$ and $g_2()$ can be found and $\Delta z = 0 \Rightarrow \Delta w = 0$ e.g. $\Delta w = a(\Delta z)^{r_1}(\ln \Delta z)^{r_2}$ Or the general problem: Given $f()$, directly or indirectly, with a singular point at z_0 say, find the functions $g_1()$ and $g_2()$ satisfying (1) or (2) or other functions defining them, for z close to z_0. Note that (2) can have z replaced by z_0 to generate an equation of the form (1) when analysing in the neighbourhood of z_0.

8 More general classes of analytic functions

Because of the elimination theorem, any algebraic function can be written with the use of redundant variables in the following form

$$P_i(z, w, x_1, \dots x_{n-1}) = 0 \text{ for } 0 \leq i \leq n \tag{94}$$

where the P_i are multivariate polynomials and the (complex) variables x_i are to be eliminated from the system resulting in a single equation of the form $P(z, w) = 0$. In few examples that I have studied, actually carrying out the stated elimination is extremely complicated and as such it may frequently be more convenient to manipulate the function in the form (94) rather than attempt the actual elimination to the form $P(z, w) = 0$ let alone the explicit algebraic formula (if it exists), using implicit function methods.

Furthermore this form suggests the extension to functions $w(z)$ defined by the following elimination problem where the P_i are polynomial functions of all their arguments:

$$P_i(z, w, x_1, \dots x_{n-1}, e^{x_1}, \dots e^{x_{n-1}}) = 0 \text{ for } 0 \leq i \leq n \tag{95}$$

may be an interesting extension of algebraic functions, regardless of whether or not such an elimination can be done explicitly. A simple example of this is the nth iterate of the exponential function which can be written in this form as

$$\begin{array}{l} x_1 - \exp(z) = 0 \\ x_2 - \exp(x_1) = 0 \\ \dots \\ x_{n-1} - \exp(x_{n-2}) = 0 \\ w - \exp(x_{n-1}) = 0 \end{array} \tag{96}$$

but not in this form for a smaller value of n showing that as the *depth* n of the system increases, more functions are included in the form (95). The depth could be defined as zero when w is expressed explicitly in terms of z by a formula.

9 Deriving the conditions for singular points in terms of derivatives of the P_i

Returning to a simpler case, suppose a analytic function $w(z)$ is is expressed implicitly by

$$P(z, w) = 0 \tag{97}$$

and even more implicitly by

$$\begin{cases} P_1(z, w, x) = 0 \\ P_2(z, w, x) = 0 \end{cases} \tag{98}$$

from which x is to be eliminated. The question is if the analytic function is defined by the form (98) how can these defining equations for singular points be expressed? One way to approach this is to write the general equations (to first order) relating the infinitesimal changes in the variables in the two different ways of expressing this relationship, and eliminate Δx from the system arising from (2) and compare it with the relationship between Δz and Δw only, arising from (1). This gives

$$\frac{\partial P}{\partial z}\Delta z + \frac{\partial P}{\partial w}\Delta w = 0 \tag{99}$$

and

$$\begin{aligned} \frac{\partial P_1}{\partial z}\Delta z + \frac{\partial P_1}{\partial w}\Delta w + \frac{\partial P_1}{\partial x}\Delta x = 0 \\ \frac{\partial P_2}{\partial z}\Delta z + \frac{\partial P_2}{\partial w}\Delta w + \frac{\partial P_2}{\partial x}\Delta x = 0 \end{aligned} \tag{100}$$

from which elimination of Δx gives

$$\Delta z \left(\frac{\partial P_2}{\partial z} - \frac{\partial P_1}{\partial z} \frac{\frac{\partial P_2}{\partial x}}{\frac{\partial P_1}{\partial x}} \right) + \Delta w \left(\frac{\partial P_2}{\partial w} - \frac{\partial P_1}{\partial w} \frac{\frac{\partial P_2}{\partial x}}{\frac{\partial P_1}{\partial x}} \right) = 0 \tag{101}$$

and comparing (99) with (101) gives

$$\frac{\partial P}{\partial z} \Big/ \left| \frac{\partial(P_1, P_2)}{\partial(x, z)} \right| = \frac{\partial P}{\partial w} \Big/ \left| \frac{\partial(P_1, P_2)}{\partial(x, w)} \right| \tag{102}$$

where the denominators are determinants of the Jacobian matrices of partial derivatives, and (6) and (7) can be represented by

$$\left| \frac{\partial(P_1, P_2)}{\partial(x, z)} \right| = 0 \tag{103}$$

and

$$\left| \frac{\partial(P_1, P_2)}{\partial(x, w)} \right| = 0 \tag{104}$$

respectively. Note that neither of these Jacobian determinants can go to infinity because the P_i and their derivatives, being polynomials, are all finite at finite values of z and w, hence finite x. Extending this argument to higher derivative conditions for singular points proved to be a little tricky.

Adding in the second order terms in the relationships amongst the infinitesimal changes to the variables, which are the leading terms omitted from (99) in the Taylor expansion of P, gives

$$\frac{\partial P}{\partial z}\Delta z + \frac{\partial P}{\partial w}\Delta w + \frac{\partial^2 P}{\partial z^2}\frac{\Delta z^2}{2} + \frac{\partial^2 P}{\partial z \partial w}\Delta z \Delta w + \frac{\partial^2 P}{\partial w^2}\frac{\Delta w^2}{2} = 0. \tag{105}$$

Likewise for (100) in the Taylor expansion of P_1 and P_2:

$$\begin{aligned} \frac{\partial P_i}{\partial z}\Delta z + \frac{\partial P_i}{\partial w}\Delta w + \frac{\partial P_i}{\partial x}\Delta x + \frac{\partial^2 P_i}{\partial z^2}\frac{\Delta z^2}{2} + \frac{\partial^2 P_i}{\partial z \partial w}\Delta z \Delta w + \frac{\partial^2 P_i}{\partial z \partial x}\Delta z \Delta x + \\ \frac{\partial^2 P_i}{\partial w \partial x}\Delta w \Delta x + \frac{\partial^2 P_i}{\partial w^2}\frac{\Delta w^2}{2} + \frac{\partial^2 P_i}{\partial x^2}\frac{\Delta x^2}{2} = 0 \text{ for } i \in \{1, 2\} \end{aligned} . \tag{106}$$

In this pair of quadratic equations for Δx, consistency requires that the linear combination of these that is linear in Δx is also satisfied. This can be written as

$$\Delta x = -\left(\frac{1}{2}\Delta z^2 a + \Delta z \Delta w B + \frac{1}{2}\Delta w^2 C + F\Delta z + G\Delta w\right)\bigg/(\Delta z D + \Delta w E + H) \tag{107}$$

where

$$A = \begin{vmatrix} \frac{\partial^2 P_1}{\partial x^2} & \frac{\partial^2 P_2}{\partial x^2} \\ \frac{\partial^2 P_1}{\partial z^2} & \frac{\partial^2 P_2}{\partial z^2} \end{vmatrix} \qquad B = \begin{vmatrix} \frac{\partial^2 P_1}{\partial x^2} & \frac{\partial^2 P_2}{\partial x^2} \\ \frac{\partial^2 P_1}{\partial z \partial w} & \frac{\partial^2 P_2}{\partial z \partial w} \end{vmatrix} \qquad C = \begin{vmatrix} \frac{\partial^2 P_1}{\partial x^2} & \frac{\partial^2 P_2}{\partial x^2} \\ \frac{\partial^2 P_1}{\partial w^2} & \frac{\partial^2 P_2}{\partial w^2} \end{vmatrix}$$
$$D = \begin{vmatrix} \frac{\partial^2 P_1}{\partial x^2} & \frac{\partial^2 P_2}{\partial x^2} \\ \frac{\partial^2 P_1}{\partial z \partial x} & \frac{\partial^2 P_2}{\partial z \partial x} \end{vmatrix} \qquad E = \begin{vmatrix} \frac{\partial^2 P_1}{\partial x^2} & \frac{\partial^2 P_2}{\partial x^2} \\ \frac{\partial^2 P_1}{\partial w \partial x} & \frac{\partial^2 P_2}{\partial w \partial x} \end{vmatrix} \qquad F = \begin{vmatrix} \frac{\partial P_1}{\partial z} & \frac{\partial P_2}{\partial z} \\ \frac{\partial^2 P_1}{\partial x^2} & \frac{\partial^2 P_2}{\partial x^2} \end{vmatrix}$$
$$G = \begin{vmatrix} \frac{\partial P_1}{\partial w} & \frac{\partial P_2}{\partial w} \\ \frac{\partial^2 P_1}{\partial x^2} & \frac{\partial^2 P_2}{\partial x^2} \end{vmatrix} \qquad H = \begin{vmatrix} \frac{\partial P_1}{\partial x} & \frac{\partial P_2}{\partial x} \\ \frac{\partial^2 P_1}{\partial x^2} & \frac{\partial^2 P_2}{\partial x^2} \end{vmatrix} \tag{108}$$

Comparing (107) with (106) from which it was derived, it is clear that (107) can be cancelled down to an expression linear in the differentials otherwise back substitution would lead to expressions involving 4th powers of Δz. It is straightforward to identify the result as

$$\Delta x = -\frac{A}{2D}\Delta z - \frac{C}{2E}\Delta w \tag{109}$$

using the highest power terms in the numerator of (107). Substituting this back into say the first of (106) (the second would give the an equivalent result because the the consistency between them, (107), has already been taken into account) gives a result of the form (105) and comparing the coefficients of the differentials in these equations shows that the following relations have to be satisfied:

$$
\begin{aligned}
\frac{\partial^2 P}{\partial z^2} &\propto \frac{\partial^2 P_1}{\partial z^2} + \frac{A^2}{4D^2}\frac{\partial^2 P_1}{\partial x^2} - \frac{A}{2D}\frac{\partial^2 P_1}{\partial z \partial x} \\
\frac{\partial^2 P}{\partial w^2} &\propto \frac{\partial^2 P_1}{\partial w^2} + \frac{C^2}{4E^2}\frac{\partial^2 P_1}{\partial x^2} - \frac{C}{2E}\frac{\partial^2 P_1}{\partial w \partial x} \\
\frac{\partial^2 P}{\partial z \partial w} &\propto \frac{\partial^2 P_1}{\partial z \partial w} + \frac{AC}{4DE}\frac{\partial^2 P_1}{\partial x^2} - \frac{A}{2D}\frac{\partial^2 P_1}{\partial w \partial x} - \frac{C}{2E}\frac{\partial^2 P_1}{\partial z \partial x} \\
\frac{\partial P}{\partial z} &\propto -\frac{A}{2D}\frac{\partial P_1}{\partial x} \\
\frac{\partial P}{\partial w} &\propto -\frac{C}{2E}\frac{\partial P_1}{\partial x}
\end{aligned}
\tag{110}
$$

where the constant of proportionality is the same for each case.

These equations are very complicated, and even more so when higher order terms are considered, so it it might be better when dealing with examples to do the eliminations to obtain Δx and (105) to obtain the coefficients which are the derivatives of P rather than using the general formulae. The suggested procedure is this: first write down the derivatives of P_i to the order needed. Do the elimination between the system (106) to obtain Δx. Substitute this back into one of (106) to obtain (105) and read off the derivatives of P needed.

How many derivatives of P are needed w.r.t. w and z? The point is to obtain all the singular points so the search must start as follows:

- Find all the points where (1) $\partial P/\partial z = 0$.
- Find all the points where (2) $\partial P/\partial w = 0$. Then for the second order derivatives:
- For each answer to (1), (1.1) find all points where also $\partial^2 P/\partial z^2 = 0$.
- For each answer to (2), (2.1) find all points where also $\partial^2 P/\partial w^2 = 0$.
- For each common answer to (1) and (2), (2.2) find all any points where also $\partial^2 P/\partial z \partial w = 0$. Then for 3rd order derivatives:
- For each answer to (1.1), find all points where also $\partial^3 P/\partial z^3 = 0$.
- For each common answer to (1.1) and (2.2) find all points where also $\partial^3 P/\partial z^2 \partial w = 0$.
- For each common answer to (2.1) and (2.2) find all points where also $\partial^3 P/\partial z \partial w^2 = 0$.
- For each answer to (2.1), find all points where also $\partial^3 P/\partial w^3 = 0$. etc..

This could be continued indefinitely and ensures that the condition attached to Equation (8) holds. The result of this search is list of all the singular points. The leading order non-zero derivatives for each such point must also be found. The values of a and r in the leading order expression $\Delta w = a\Delta z^r$ can then be obtained [Nixon2013] for each singular point.

Given all the pairs of values of a and r for a singular point at (z_0, w_0) can the leading order non-zero derivatives of P at (z_0, w_0) be obtained?

References

[Nixon2013] Nixon J., Theory of algebraic functions on the Riemann Sphere Mathematica Aeterna Vol. 3, 2013, no. 2, 83-101 https://www.longdom.org/articles/theory-of-algebraic-functions-on-the-riemann-sphere.pdf