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Turing Machines

Abstract

to be added

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Keywords: Turing machine Comments are welcome. Please send them to john.h.nixon1@gmail.com

This document is a work in progress. As such it is incomplete and still has errors and omissions. When brought to a state where I cannot easily find any improvements it will form my next paper on Turing Machine analysis.

Theorem 4.1 has been added and this will remove some unnecessary duplication. The worked examples in section 7 need to be looked at again as do the lists of results in section 8.

A lot of the old material is now after the references so it can be viewed with links that still work but will probably not be in the paper.

This version of the paper has a new version of the program linked to it [6], (the old version is [3]). It now creates Table 6 correctly. Note that the numbering of the IRR outlines is different from the paper. For values between 1 and 68: add 132 to get program output. For values between 1 and 133 inclusive: subtract 1. The reason for this is that the Tables 2 and the equivalent with the dots on the left were calculated together in the program.

The final section on reverse rules might only be needed in a simplified form. A lot of material has been removed to 2017's Notes on Turing Machines. These notes are now mostly superseded, but there may be a little there that is of use.

1 The (5,5) TM and the first steps for its analysis: generating its irreducible regular rules (IRR) up to length 3

This document describes my analysis techniques applied to the following Turing Machine

$$
\begin{array}{lllll}
1\underline{a} \to 2\_d & 2\underline{a} \to 1c\_ & 3\underline{a} \to 4c\_ & 4\underline{a} \to 3\_b & 5\underline{a} \to 2\_e \\
1\underline{b} \to 4\_d & 2\underline{b} \to 4\_c & 3\underline{b} \to 4\_c & 4\underline{b} \to 4b\_ & 5\underline{b} \to 3\_e \\
1\underline{c} \to 3\_a & 2\underline{c} \to 1d\_ & 3\underline{c} \to 2\_a & 4\underline{c} \to 3c\_ & 5\underline{c} \to 3a\_ \\
1\underline{d} \to 2b\_ & 2\underline{d} \to 1a\_ & 3\underline{d} \to 5\_c & 4\underline{d} \to 5\_c & 5\underline{d} \to 4\_a \\
1\underline{e} \to 2b\_ & 2\underline{e} \to 3\_c & 3\underline{e} \to 3b\_ & 4\underline{e} \to 5a\_ & 5\underline{e} \to 3a\_
\end{array}
\tag{1}
$$

(TM) which was generated randomly with 5 states and 5 symbols. This TM, being much larger than any that I have analysed before, has proved to be a much more challenging case and will help me refine the procedures for efficiently generating the irreducible regular rules (IRR) from a TM that allowed understandable descriptions of the behaviour of smaller TM's to be found.

By looking for backward TM steps as described previously in [2], the following summary is obtained:

$$
\begin{array}{ll}
\mathtt{1\_}\alpha \leftarrow \emptyset & \mathtt{1}\alpha\mathtt{\_} \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} \mathtt{2\underline{d}} \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{2\underline{a}} \\ \overset{\alpha=\mathtt{d}}{\leftarrow} \mathtt{2\underline{c}} \end{cases} \\
\mathtt{2\_}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} \mathtt{3\underline{c}} \\ \overset{\alpha=\mathtt{d}}{\leftarrow} \mathtt{1\underline{a}} \\ \overset{\alpha=\mathtt{e}}{\leftarrow} \mathtt{5\underline{a}} \end{cases} & \mathtt{2}\alpha\mathtt{\_} \overset{\alpha=\mathtt{b}}{\leftarrow} \begin{cases} \mathtt{1\underline{d}} \\ \mathtt{1\underline{e}} \end{cases} \\
\mathtt{3\_}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} \mathtt{1\underline{c}} \\ \overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{4\underline{a}} \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{2\underline{e}} \\ \overset{\alpha=\mathtt{e}}{\leftarrow} \mathtt{5\underline{b}} \end{cases} & \mathtt{3}\alpha\mathtt{\_} \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} \begin{cases} \mathtt{5\underline{c}} \\ \mathtt{5\underline{e}} \end{cases} \\ \overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{3\underline{e}} \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{4\underline{c}} \end{cases} \\
\mathtt{4\_}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} \mathtt{5\underline{d}} \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \begin{cases} \mathtt{2\underline{b}} \\ \mathtt{3\underline{b}} \end{cases} \\ \overset{\alpha=\mathtt{d}}{\leftarrow} \mathtt{1\underline{b}} \end{cases} & \mathtt{4}\alpha\mathtt{\_} \begin{cases} \overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{4\underline{b}} \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{3\underline{a}} \end{cases} \\
\mathtt{5\_}\alpha \overset{\alpha=\mathtt{c}}{\leftarrow} \begin{cases} \mathtt{3\underline{d}} \\ \mathtt{4\underline{d}} \end{cases} & \mathtt{5}\alpha\mathtt{\_} \overset{\alpha=\mathtt{a}}{\leftarrow} \mathtt{4\underline{e}}
\end{array} \tag{2}
$$

By adding an arbitrary symbol at the pointer to the LHS's of (2), reachability (in the sense described in [2]) of these can be assured, but do these generate IRR's of length 2? This is true if the forward computation has as its first step a move to the other position with a symbol rather than going beyond the string. This ensures that this forward computation generates an irreducible rule regardless of whether the pointer ends up at the left or right. This condition determines the set of symbols α_2 in each case in the following.

$$\mathtt{1}\alpha_1\underline{\alpha_2}\begin{cases} \overset{\alpha_1=\mathtt{a}}{\longleftarrow} \mathtt{2\underline{d}}\alpha_2 \\ \overset{\alpha_1=\mathtt{c}}{\longleftarrow} \mathtt{2\underline{a}}\alpha_2 \\ \overset{\alpha_1=\mathtt{d}}{\longleftarrow} \mathtt{2\underline{c}}\alpha_2 \end{cases} \text{for } \alpha_2 \in \{\mathtt{a},\mathtt{b},\mathtt{c}\} \qquad \mathtt{2}\underline{\alpha_2}\alpha_1\begin{cases} \overset{\alpha_1=\mathtt{a}}{\longleftarrow} \mathtt{3}\alpha_2\mathtt{\underline{c}} \\ \overset{\alpha_1=\mathtt{d}}{\longleftarrow} \mathtt{1}\alpha_2\mathtt{\underline{a}} \\ \overset{\alpha_1=\mathtt{e}}{\longleftarrow} \mathtt{5}\alpha_2\mathtt{\underline{a}} \end{cases} \text{for } \alpha_2 \in \{\mathtt{a},\mathtt{c},\mathtt{d}\}$$

$$\mathtt{2b}\underline{\alpha_2} \leftarrow \begin{cases} \mathtt{1\underline{d}}\alpha_2 \\ \mathtt{1\underline{e}}\alpha_2 \end{cases} \text{for } \alpha_2 \in \{\mathtt{b},\mathtt{e}\} \qquad \mathtt{3}\underline{\alpha_2}\alpha_1\begin{cases} \overset{\alpha_1=\mathtt{a}}{\longleftarrow} \mathtt{1}\alpha_2\mathtt{\underline{c}} \\ \overset{\alpha_1=\mathtt{b}}{\longleftarrow} \mathtt{4}\alpha_2\mathtt{\underline{a}} \\ \overset{\alpha_1=\mathtt{c}}{\longleftarrow} \mathtt{2}\alpha_2\mathtt{\underline{e}} \\ \overset{\alpha_1=\mathtt{e}}{\longleftarrow} \mathtt{5}\alpha_2\mathtt{\underline{b}} \end{cases} \text{for } \alpha_2 \in \{\mathtt{a},\mathtt{e}\}$$

$$\mathtt{3}\alpha_1\underline{\alpha_2}\begin{cases} \overset{\alpha_1=\mathtt{a}}{\longleftarrow} \begin{cases} \mathtt{5\underline{c}}\alpha_2 \\ \mathtt{5\underline{e}}\alpha_2 \end{cases} \\ \overset{\alpha_1=\mathtt{b}}{\longleftarrow} \mathtt{3\underline{e}}\alpha_2 \\ \overset{\alpha_1=\mathtt{c}}{\longleftarrow} \mathtt{4\underline{c}}\alpha_2 \end{cases} \text{for } \alpha_2 \in \{\mathtt{b},\mathtt{c},\mathtt{d}\} \qquad \mathtt{4}\underline{\alpha_2}\alpha_1\begin{cases} \overset{\alpha_1=\mathtt{a}}{\longleftarrow} \mathtt{5}\alpha_2\mathtt{\underline{d}} \\ \overset{\alpha_1=\mathtt{c}}{\longleftarrow} \begin{cases} \mathtt{2}\alpha_2\mathtt{\underline{b}} \\ \mathtt{3}\alpha_2\mathtt{\underline{b}} \end{cases} \\ \overset{\alpha_1=\mathtt{d}}{\longleftarrow} \mathtt{1}\alpha_2\mathtt{\underline{b}} \end{cases} \text{for } \alpha_2 \in \{\mathtt{b},\mathtt{c},\mathtt{e}\}$$

$$\mathtt{4}\alpha_1\underline{\alpha_2}\begin{cases} \overset{\alpha_1=\mathtt{b}}{\longleftarrow} \mathtt{4\underline{b}}\alpha_2 \\ \overset{\alpha_1=\mathtt{c}}{\longleftarrow} \mathtt{3\underline{a}}\alpha_2 \end{cases} \text{for } \alpha_2 \in \{\mathtt{a},\mathtt{d}\} \qquad \mathtt{5}\underline{\alpha_2}\mathtt{c} \leftarrow \begin{cases} \mathtt{3}\alpha_2\mathtt{\underline{d}} \\ \mathtt{4}\alpha_2\mathtt{\underline{d}} \end{cases} \text{for } \alpha_2 \in \{\mathtt{c},\mathtt{e}\}$$

$$\mathtt{5a}\underline{\alpha_2} \leftarrow \mathtt{4\underline{e}}\alpha_2 \text{ for } \alpha_2 \in \{\mathtt{a},\mathtt{b},\mathtt{d}\} \tag{3}$$

The irreducible forward computations referred to above are as follows. These are therefore the IRR of length 2 for TM1 defined in (1).

$$\begin{array}{lllll}
\mathtt{1a\underline{a}} \to \mathtt{2cb_} & \mathtt{1c\underline{a}} \to \mathtt{2db_} & \mathtt{1d\underline{a}} \to \mathtt{2ab_} & \mathtt{1a\underline{b}} \to \mathtt{3_bd} & \mathtt{1c\underline{b}} \to \mathtt{2db_} \\
\mathtt{1d\underline{b}} \to \mathtt{5_cd} & \mathtt{1a\underline{c}} \to \mathtt{2_ab} & \mathtt{1c\underline{c}} \to \mathtt{2_aa} & \mathtt{1d\underline{c}} \to \mathtt{5_ca} & \mathtt{2\underline{a}a} \to \mathtt{2db_} \\
\mathtt{2\underline{a}d} \to \mathtt{2cb_} & \mathtt{2\underline{a}e} \to \mathtt{2cb_} & \mathtt{2\underline{c}a} \to \mathtt{2ab_} & \mathtt{2\underline{c}d} \to \mathtt{2db_} & \mathtt{2\underline{c}e} \to \mathtt{2db_} \\
\mathtt{2\underline{d}a} \to \mathtt{2cb_} & \mathtt{2\underline{d}d} \to \mathtt{2ab_} & \mathtt{2\underline{d}e} \to \mathtt{2ab_} & \mathtt{2b\underline{b}} \to \mathtt{3bc_} & \mathtt{2b\underline{e}} \to \mathtt{4_cc} \\
\mathtt{3\underline{a}a} \to \mathtt{2_ab} & \mathtt{3\underline{a}b} \to \mathtt{4cb_} & \mathtt{3\underline{a}c} \to \mathtt{3cc_} & \mathtt{3\underline{a}e} \to \mathtt{5ca_} & \mathtt{3\underline{e}a} \to \mathtt{4bc_} \\
\mathtt{3\underline{e}b} \to \mathtt{3bc_} & \mathtt{3\underline{e}c} \to \mathtt{4_ca} & \mathtt{3\underline{e}e} \to \mathtt{3bb_} & \mathtt{3a\underline{b}} \to \mathtt{3_bc} & \mathtt{3b\underline{b}} \to \mathtt{3bc_} \\
\mathtt{3c\underline{b}} \to \mathtt{2ab_} & \mathtt{3a\underline{c}} \to \mathtt{2db_} & \mathtt{3b\underline{c}} \to \mathtt{4_ca} & \mathtt{3c\underline{c}} \to \mathtt{2ab_} & \mathtt{3a\underline{d}} \to \mathtt{2_ec} \\
\mathtt{3b\underline{d}} \to \mathtt{3_ec} & \mathtt{3c\underline{d}} \to \mathtt{2db_} & \mathtt{4\underline{b}a} \to \mathtt{4_cb} & \mathtt{4\underline{b}c} \to \mathtt{3bc_} & \mathtt{4\underline{b}d} \to \mathtt{3_ec} \\
\mathtt{4\underline{c}a} \to \mathtt{4cc_} & \mathtt{4\underline{c}c} \to \mathtt{2ab_} & \mathtt{4\underline{c}d} \to \mathtt{2db_} & \mathtt{4\underline{e}a} \to \mathtt{2cb_} & \mathtt{4\underline{e}c} \to \mathtt{3aa_} \\
\mathtt{4\underline{e}d} \to \mathtt{3_ba} & \mathtt{4b\underline{a}} \to \mathtt{4_cb} & \mathtt{4c\underline{a}} \to \mathtt{2_ab} & \mathtt{4b\underline{d}} \to \mathtt{3_ec} & \mathtt{4c\underline{d}} \to \mathtt{2db_} \\
\mathtt{5\underline{c}c} \to \mathtt{2db_} & \mathtt{5\underline{e}c} \to \mathtt{2db_} & \mathtt{5a\underline{a}} \to \mathtt{2cb_} & \mathtt{5a\underline{b}} \to \mathtt{5ca_} & \mathtt{5a\underline{d}} \to \mathtt{3_ba}
\end{array} \tag{4}$$

Taking the first case in (3) namely $\mathtt{1a}\underline{\alpha_2} \leftarrow \mathtt{2\underline{d}}\alpha_2$ and continuing the calculation forward as in (4) gives

$$\begin{array}{l}
\mathtt{2\underline{d}a} \to \mathtt{1a\underline{a}} \to \mathtt{2cb_} \\
\mathtt{2\underline{d}b} \to \mathtt{1a\underline{b}} \to \mathtt{3_bd} \\
\mathtt{2\underline{d}c} \to \mathtt{1a\underline{c}} \to \mathtt{2_ab}
\end{array} \tag{5}$$

having type RR, RL and RL respectively, showing that for this case $\alpha_2 \in \{\mathtt{b},\mathtt{c}\}$ is necessary for these IRR to be extendable to length 3. Doing so also needs the addition of an extra arbitrary symbol say α_3 on the left (because the pointer is at the left in the LHS's of (5)) and using the backward search algorithm to determine for which values of α_3 the LHS is reachable. This gives a result which amounts to using the reverse rule

$$\mathtt{2}\alpha_3\underline{\mathtt{d}}\alpha_2 \begin{cases} \overset{\alpha_3=\mathtt{b}}{\longleftarrow} \begin{cases} \mathtt{1}\underline{\mathtt{d}}\mathtt{d}\alpha_2 \\ \mathtt{1}\underline{\mathtt{e}}\mathtt{d}\alpha_2 \end{cases} \\ \overset{\alpha_2=\mathtt{a}}{\longleftarrow} \mathtt{3}\alpha_3\mathtt{d}\underline{\mathtt{c}} \\ \overset{\alpha_2=\mathtt{d}}{\longleftarrow} \mathtt{1}\alpha_3\mathtt{d}\underline{\mathtt{a}} \\ \overset{\alpha_2=\mathtt{e}}{\longleftarrow} \mathtt{5}\alpha_3\mathtt{d}\underline{\mathtt{a}} \end{cases} . \tag{6}$$

The above restriction on α_2 eliminates the bottom 3 branches, so for an origin to exist needs $\alpha_3 = \mathtt{b}$. So this gives the following results for the corresponding IRR(3)

$$\left.\begin{array}{l} \mathtt{1}\underline{\mathtt{d}}\mathtt{d}\alpha_2 \\ \mathtt{1}\underline{\mathtt{e}}\mathtt{d}\alpha_2 \end{array}\right\} \to \mathtt{1ba}\underline{\alpha_2} \begin{cases} \overset{\alpha_2=\mathtt{b}}{\longrightarrow} \mathtt{4\_cbd} \\ \overset{\alpha_2=\mathtt{c}}{\longrightarrow} \mathtt{4\_cab} \end{cases} . \tag{7}$$

(Note that in all the calculations that follow, the IRR will be indicated by 3 parts namely the origin, LHS and RHS, which for reasons of compatibility with later terminology will be called respectively $\mathtt{A}$, $\mathtt{B}$ and $\mathtt{C}$ and the the IRR is $\mathtt{A} \to \mathtt{B} \to \mathtt{C}$.)

This procedure was repeated starting from each origin in (3) and gives the following set reverse rules.

$$\mathtt{1}\alpha_3\underline{\mathtt{d}}\mathtt{e} \begin{cases} \overset{\alpha_3=\mathtt{a}}{\longleftarrow} \mathtt{2}\underline{\mathtt{d}}\mathtt{de} \\ \overset{\alpha_3=\mathtt{c}}{\longleftarrow} \mathtt{2}\underline{\mathtt{a}}\mathtt{de} \\ \overset{\alpha_3=\mathtt{d}}{\longleftarrow} \mathtt{2}\underline{\mathtt{c}}\mathtt{de} \end{cases} \tag{8}$$

$$\mathtt{1}\alpha_3\underline{\mathtt{e}}\mathtt{e} \begin{cases} \overset{\alpha_3=\mathtt{a}}{\longleftarrow} \mathtt{2}\underline{\mathtt{d}}\mathtt{ee} \\ \overset{\alpha_3=\mathtt{c}}{\longleftarrow} \mathtt{2}\underline{\mathtt{a}}\mathtt{ee} \\ \overset{\alpha_3=\mathtt{d}}{\longleftarrow} \mathtt{2}\underline{\mathtt{c}}\mathtt{ee} \end{cases} \tag{9}$$

$$\mathtt{1}\alpha_2\underline{\mathtt{a}}\alpha_3 \leftarrow \begin{cases} \overset{\alpha_2=\mathtt{a}}{\longleftarrow} \mathtt{2}\underline{\mathtt{d}}\mathtt{a}\alpha_3 \\ \overset{\alpha_2=\mathtt{c}}{\longleftarrow} \mathtt{2}\underline{\mathtt{a}}\mathtt{a}\alpha_3 \\ \overset{\alpha_2=\mathtt{d}}{\longleftarrow} \mathtt{2}\underline{\mathtt{c}}\mathtt{a}\alpha_3 \end{cases} \quad \text{for } \alpha_2 \in \{\mathtt{a}, \mathtt{c}, \mathtt{d}\} \tag{10}$$

$$\mathtt{1c}\underline{\mathtt{b}}\alpha_3 \leftarrow \mathtt{2}\underline{\mathtt{a}}\mathtt{b}\alpha_3 \tag{11}$$

$$\mathtt{1e}\underline{\mathtt{c}}\alpha_3 \leftarrow \emptyset \tag{12}$$

$$\mathtt{2}\alpha_3\underline{\mathtt{a}}\mathtt{c} \overset{\alpha_3=\mathtt{b}}{\longleftarrow} \begin{cases} \mathtt{1}\underline{\mathtt{d}}\mathtt{ac} \\ \mathtt{1}\underline{\mathtt{e}}\mathtt{ac} \end{cases} \tag{13}$$

$$2\alpha_3\underline{\mathtt{c}}\alpha_2 \begin{cases} \xleftarrow{\alpha_3=\mathtt{b}} \begin{cases} 1\underline{\mathtt{d}}\mathtt{c}\alpha_2 \\ 1\underline{\mathtt{e}}\mathtt{c}\alpha_2 \end{cases} \\ \xleftarrow{\alpha_2=\mathtt{a}} 3\alpha_3\mathtt{c}\underline{\mathtt{c}} \end{cases} \quad \text{for } \alpha_2 \in \{\mathtt{a},\mathtt{b},\mathtt{c}\} \tag{14}$$

$$2\alpha_3\underline{\mathtt{d}}\alpha_2 \begin{cases} \xleftarrow{\alpha_3=\mathtt{b}} \begin{cases} 1\underline{\mathtt{d}}\mathtt{d}\alpha_2 \\ 1\underline{\mathtt{e}}\mathtt{d}\alpha_2 \end{cases} \\ \xleftarrow{\alpha_2=\mathtt{a}} 3\alpha_3\mathtt{d}\underline{\mathtt{c}} \end{cases} \quad \text{for } \alpha_2 \in \{\mathtt{a},\mathtt{b},\mathtt{c}\}. \tag{15}$$

$$2\alpha_2\underline{\mathtt{b}}\alpha_3 \begin{cases} \xleftarrow{\alpha_3=\mathtt{a}} 3\alpha_2\mathtt{b}\underline{\mathtt{c}} \\ \xleftarrow{\alpha_3=\mathtt{d}} 1\alpha_2\mathtt{b}\underline{\mathtt{a}} \\ \xleftarrow{\alpha_3=\mathtt{e}} 5\alpha_2\mathtt{b}\underline{\mathtt{a}} \end{cases} \quad \text{for } \alpha_2 \in \{\mathtt{b},\mathtt{c},\mathtt{e}\} \tag{16}$$

$$2\mathtt{a}\underline{\mathtt{e}}\alpha_3 \begin{cases} \xleftarrow{\alpha_3=\mathtt{a}} 3\mathtt{ae}\underline{\mathtt{c}} \\ \xleftarrow{\alpha_3=\mathtt{d}} 1\mathtt{ae}\underline{\mathtt{a}} \\ \xleftarrow{\alpha_3=\mathtt{e}} 5\mathtt{ea}\underline{\mathtt{a}} \end{cases} \tag{17}$$

$$3\alpha_3\underline{\mathtt{a}}\mathtt{a} \begin{cases} \xleftarrow{\alpha_3=\mathtt{a}} \begin{cases} 5\underline{\mathtt{c}}\mathtt{aa} \\ 5\underline{\mathtt{e}}\mathtt{aa} \end{cases} \\ \xleftarrow{\alpha_3=\mathtt{b}} 3\underline{\mathtt{e}}\mathtt{aa} \\ \xleftarrow{\alpha_3=\mathtt{c}} 4\underline{\mathtt{c}}\mathtt{aa} \end{cases} \tag{18}$$

$$3\alpha_3\underline{\mathtt{e}}\alpha_2 \begin{cases} \xleftarrow{\alpha_3=\mathtt{a}} \begin{cases} 5\underline{\mathtt{c}}\mathtt{e}\alpha_2 \\ 5\underline{\mathtt{e}}\mathtt{e}\alpha_2 \end{cases} \\ \xleftarrow{\alpha_3=\mathtt{b}} 3\underline{\mathtt{e}}\mathtt{e}\alpha_2 \\ \xleftarrow{\alpha_3=\mathtt{c}} 4\underline{\mathtt{c}}\mathtt{e}\alpha_2 \end{cases} \quad \text{for } \alpha_2 \in \{\mathtt{c},\mathtt{d}\} \tag{19}$$

$$3\alpha_2\underline{\mathtt{b}}\alpha_3 \begin{cases} \xleftarrow{\alpha_3=\mathtt{a}} 1\alpha_2\mathtt{b}\underline{\mathtt{c}} \\ \xleftarrow{\alpha_3=\mathtt{b}} 4\alpha_2\mathtt{b}\underline{\mathtt{a}} \\ \xleftarrow{\alpha_3=\mathtt{c}} 2\alpha_2\mathtt{b}\underline{\mathtt{e}} \\ \xleftarrow{\alpha_3=\mathtt{e}} 5\alpha_2\mathtt{b}\underline{\mathtt{b}} \end{cases} \quad \text{for } \alpha_2 \in \{\mathtt{b},\mathtt{c},\mathtt{e}\} \tag{20}$$

$$3\alpha_2\underline{\mathtt{c}}\alpha_3 \leftarrow \begin{cases} \xleftarrow{\alpha_3=\mathtt{a}} 1\alpha_2\mathtt{c}\underline{\mathtt{c}} \\ \xleftarrow{\alpha_3=\mathtt{b}} 4\alpha_2\mathtt{c}\underline{\mathtt{a}} \\ \xleftarrow{\alpha_3=\mathtt{c}} 2\alpha_2\mathtt{c}\underline{\mathtt{e}} \\ \xleftarrow{\alpha_3=\mathtt{e}} 5\alpha_2\mathtt{c}\underline{\mathtt{b}} \\ \xleftarrow{\alpha_2=\mathtt{a}} \begin{cases} 5\underline{\mathtt{c}}\mathtt{c}\alpha_3 \\ 5\underline{\mathtt{e}}\mathtt{c}\alpha_3 \end{cases} \\ \xleftarrow{\alpha_2=\mathtt{c}} 4\underline{\mathtt{c}}\mathtt{c}\alpha_3 \end{cases} \quad \text{for } \alpha_2 \in \{\mathtt{a},\mathtt{c},\mathtt{d}\} \tag{21}$$

$$3\alpha_2\underline{\mathtt{d}}\alpha_3 \begin{cases} \xleftarrow{\alpha_3=\mathtt{a}} 1\alpha_2\mathtt{d}\underline{\mathtt{c}} \\ \xleftarrow{\alpha_3=\mathtt{b}} 4\alpha_2\mathtt{d}\underline{\mathtt{a}} \\ \xleftarrow{\alpha_3=\mathtt{c}} 2\alpha_2\mathtt{d}\underline{\mathtt{e}} \\ \xleftarrow{\alpha_3=\mathtt{e}} 5\alpha_2\mathtt{d}\underline{\mathtt{b}} \end{cases} \quad \text{for } \alpha_2 \in \{\mathtt{d},\mathtt{e}\} \tag{22}$$

$$4\alpha_3\underline{\mathtt{b}}\alpha_2\begin{cases}\xleftarrow{\alpha_3=\mathtt{b}} 4\underline{\mathtt{b}}\mathtt{b}\alpha_2\\ \xleftarrow{\alpha_3=\mathtt{c}} 3\underline{\mathtt{a}}\mathtt{b}\alpha_2\end{cases}\quad \text{for } \alpha_2\in\{\mathtt{a},\mathtt{d}\} \tag{23}$$

$$4\alpha_3\underline{\mathtt{e}}\mathtt{d}\begin{cases}\xleftarrow{\alpha_3=\mathtt{b}} 4\underline{\mathtt{b}}\mathtt{ed}\\ \xleftarrow{\alpha_3=\mathtt{c}} 3\underline{\mathtt{a}}\mathtt{ed}\end{cases} \tag{24}$$

$$4\mathtt{a}\underline{\mathtt{a}}\alpha_3\begin{cases}\xleftarrow{\alpha_3=\mathtt{a}} 5\mathtt{aa}\underline{\mathtt{d}}\\ \xleftarrow{\alpha_3=\mathtt{c}} \begin{cases}2\mathtt{aa}\underline{\mathtt{b}}\\ 3\mathtt{aa}\underline{\mathtt{b}}\end{cases}\\ \xleftarrow{\alpha_3=\mathtt{d}} 1\mathtt{aa}\underline{\mathtt{b}}\end{cases} \tag{25}$$

$$4\mathtt{e}\underline{\mathtt{a}}\alpha_3\begin{cases}\xleftarrow{\alpha_3=\mathtt{a}} 5\mathtt{ea}\underline{\mathtt{d}}\\ \xleftarrow{\alpha_3=\mathtt{c}} \begin{cases}2\mathtt{ea}\underline{\mathtt{b}}\\ 3\mathtt{ea}\underline{\mathtt{b}}\end{cases}\\ \xleftarrow{\alpha_3=\mathtt{d}} 1\mathtt{ea}\underline{\mathtt{b}}\end{cases} \tag{26}$$

$$4\alpha_2\underline{\mathtt{d}}\alpha_3\begin{cases}\xleftarrow{\alpha_3=\mathtt{a}} 5\alpha_2\mathtt{d}\underline{\mathtt{d}}\\ \xleftarrow{\alpha_3=\mathtt{c}} \begin{cases}2\alpha_2\mathtt{d}\underline{\mathtt{b}}\\ 3\alpha_2\mathtt{d}\underline{\mathtt{b}}\end{cases}\\ \xleftarrow{\alpha_3=\mathtt{d}} 1\alpha_2\mathtt{d}\underline{\mathtt{b}}\\ \xleftarrow{\alpha_2=\mathtt{c}} 3\underline{\mathtt{a}}\mathtt{d}\alpha_3\end{cases}\quad \text{for } \alpha_2\in\{\mathtt{c},\mathtt{e}\} \tag{27}$$

$$5\alpha_3\underline{\mathtt{c}}\mathtt{b}\xleftarrow{\alpha_3=\mathtt{a}} 4\underline{\mathtt{e}}\mathtt{cb} \tag{28}$$

$$5\alpha_3\underline{\mathtt{c}}\mathtt{d}\xleftarrow{\alpha_3=\mathtt{a}} 4\underline{\mathtt{e}}\mathtt{cd} \tag{29}$$

$$5\alpha_3\underline{\mathtt{e}}\mathtt{b}\xleftarrow{\alpha_3=\mathtt{a}} 4\underline{\mathtt{e}}\mathtt{eb} \tag{30}$$

$$5\alpha_3\underline{\mathtt{e}}\mathtt{d}\xleftarrow{\alpha_3=\mathtt{a}} 4\underline{\mathtt{e}}\mathtt{ed} \tag{31}$$

$$5\alpha_2\underline{\mathtt{a}}\alpha_3\begin{cases}\xleftarrow{\alpha_3=\mathtt{c}} \begin{cases}3\alpha_2\mathtt{a}\underline{\mathtt{d}}\\ 4\alpha_2\mathtt{a}\underline{\mathtt{d}}\end{cases}\\ \xleftarrow{\alpha_2=\mathtt{a}} 4\underline{\mathtt{e}}\mathtt{a}\alpha_3\end{cases}\quad \text{for } \alpha_2\in\{\mathtt{a},\mathtt{c},\mathtt{d}\} \tag{32}$$

$$5\mathtt{a}\underline{\mathtt{b}}\alpha_3\xleftarrow{\alpha_3=\mathtt{c}}\begin{cases}3\mathtt{ab}\underline{\mathtt{d}}\\ 4\mathtt{ab}\underline{\mathtt{d}}\end{cases} \tag{33}$$

$$5\mathtt{e}\underline{\mathtt{b}}\alpha_3\xleftarrow{\alpha_3=\mathtt{c}}\begin{cases}3\mathtt{eb}\underline{\mathtt{d}}\\ 4\mathtt{eb}\underline{\mathtt{d}}\end{cases} \tag{34}$$

$$5\alpha_2\underline{\mathtt{d}}\alpha_3 \stackrel{\alpha_3=\mathtt{c}}{\longleftarrow} \left\{ \begin{array}{l} 3\alpha_2\mathtt{d}\underline{\mathtt{d}} \\ 4\alpha_2\mathtt{d}\underline{\mathtt{d}} \end{array} \right. \quad \text{for } \alpha_2 \in \{\mathtt{c}, \mathtt{e}\} \tag{35}$$

The following is the set of IRR(3) derived.

$$\left. \begin{array}{l} 1\underline{\mathtt{d}}\mathtt{db} \\ 1\underline{\mathtt{e}}\mathtt{db} \end{array} \right\} \to 1\mathtt{ba}\underline{\mathtt{b}} \to 4\_\mathtt{cbd} \tag{36}$$

$$\left. \begin{array}{l} 1\underline{\mathtt{d}}\mathtt{ac} \\ 1\underline{\mathtt{e}}\mathtt{ac} \end{array} \right\} \to 1\mathtt{bc}\underline{\mathtt{c}} \to 4\_\mathtt{caa} \tag{37}$$

$$\left. \begin{array}{l} 1\underline{\mathtt{d}}\mathtt{c}\alpha_2 \\ 1\underline{\mathtt{e}}\mathtt{c}\alpha_2 \end{array} \right\} \to 1\mathtt{bd}\underline{\alpha_2} \left\{ \begin{array}{l} \stackrel{\alpha_2=\mathtt{b}}{\longrightarrow} 3\_\mathtt{ecd} \\ \stackrel{\alpha_2=\mathtt{c}}{\longrightarrow} 3\_\mathtt{eca} \end{array} \right. \tag{38}$$

$$\left. \begin{array}{l} 1\underline{\mathtt{d}}\mathtt{dc} \\ 1\underline{\mathtt{e}}\mathtt{dc} \end{array} \right\} \to 1\mathtt{ba}\underline{\mathtt{c}} \to 4\_\mathtt{cab} \tag{39}$$

$$1\alpha_2\mathtt{c}\underline{\mathtt{c}} \left\{ \begin{array}{l} \stackrel{\alpha_2=\mathtt{a}}{\longrightarrow} 2\underline{\mathtt{a}}\mathtt{aa} \to 1\mathtt{dbc}\_ \\ \stackrel{\alpha_2=\mathtt{c}}{\longrightarrow} 2\underline{\mathtt{c}}\mathtt{aa} \to 1\mathtt{abc}\_ \\ \stackrel{\alpha_2=\mathtt{d}}{\longrightarrow} 2\underline{\mathtt{d}}\mathtt{aa} \to 1\mathtt{cbc}\_ \end{array} \right. \tag{40}$$

$$4\alpha_2\mathtt{c}\underline{\mathtt{a}} \left\{ \begin{array}{l} \stackrel{\alpha_2=\mathtt{a}}{\longrightarrow} 2\underline{\mathtt{a}}\mathtt{ab} \to 3\mathtt{dbc}\_ \\ \stackrel{\alpha_2=\mathtt{c}}{\longrightarrow} 2\underline{\mathtt{c}}\mathtt{ab} \to 3\mathtt{abc}\_ \\ \stackrel{\alpha_2=\mathtt{d}}{\longrightarrow} 2\underline{\mathtt{d}}\mathtt{ab} \to 3\mathtt{cbc}\_ \end{array} \right. \tag{41}$$

$$2\alpha_2\mathtt{c}\underline{\mathtt{e}} \left\{ \begin{array}{l} \stackrel{\alpha_2=\mathtt{a}}{\longrightarrow} 2\underline{\mathtt{a}}\mathtt{ac} \to 1\mathtt{dbd}\_ \\ \stackrel{\alpha_2=\mathtt{c}}{\longrightarrow} 2\underline{\mathtt{c}}\mathtt{ac} \to 1\mathtt{abd}\_ \\ \stackrel{\alpha_2=\mathtt{d}}{\longrightarrow} 2\underline{\mathtt{d}}\mathtt{ac} \to 1\mathtt{cbd}\_ \end{array} \right. \tag{42}$$

$$5\alpha_2\mathtt{c}\underline{\mathtt{b}} \left\{ \begin{array}{l} \stackrel{\alpha_2=\mathtt{a}}{\longrightarrow} 2\underline{\mathtt{a}}\mathtt{ae} \to 5\_\mathtt{ccc} \\ \stackrel{\alpha_2=\mathtt{c}}{\longrightarrow} 2\underline{\mathtt{c}}\mathtt{ae} \to 3\_\mathtt{bcc} \\ \stackrel{\alpha_2=\mathtt{d}}{\longrightarrow} 2\underline{\mathtt{d}}\mathtt{ae} \to 1\mathtt{abd}\_ \end{array} \right. \tag{43}$$

$$\left. \begin{array}{l} 3\alpha_2\mathtt{a}\underline{\mathtt{d}} \\ 4\alpha_2\mathtt{a}\underline{\mathtt{d}} \end{array} \right\} \to 2\underline{\alpha_2}\mathtt{ec} \left\{ \begin{array}{l} \stackrel{\alpha_2=\mathtt{a}}{\longrightarrow} 1\mathtt{cbd}\_ \\ \stackrel{\alpha_2=\mathtt{c}}{\longrightarrow} 1\mathtt{dbd}\_ \\ \stackrel{\alpha_2=\mathtt{d}}{\longrightarrow} 1\mathtt{abd}\_ \end{array} \right. \tag{44}$$

$$2\underline{\mathtt{d}} \left\{ \begin{array}{c} \mathtt{d} \\ \mathtt{e} \end{array} \right\} \mathtt{e} \to 2\mathtt{ab}\underline{\mathtt{e}} \to 3\_\mathtt{bcc} \tag{45}$$

$$2\underline{\mathtt{a}} \left\{ \begin{array}{c} \mathtt{d} \\ \mathtt{e} \end{array} \right\} \mathtt{e} \to 2\mathtt{cb}\underline{\mathtt{e}} \to 1\mathtt{abd}\_ \tag{46}$$

$$\texttt{2\underline{c}}\left\{\begin{array}{c}\texttt{d}\\ \texttt{e}\end{array}\right\}\texttt{e} \to \texttt{2db\underline{e}} \to \texttt{5\_ccc} \tag{47}$$

$$\texttt{5aa\underline{d}} \to \texttt{3\underline{a}ba} \to \texttt{3abc\_} \tag{48}$$

$$\left.\begin{array}{l}\texttt{2aa\underline{b}}\\ \texttt{3aa\underline{b}}\end{array}\right\} \to \texttt{3\underline{a}bc} \to \texttt{3cbc\_} \tag{49}$$

$$\texttt{1aa\underline{b}} \to \texttt{3\underline{a}bd} \to \texttt{2\_aec} \tag{50}$$

$$\texttt{5ea\underline{d}} \to \texttt{3\underline{e}ba} \to \texttt{4bcc\_} \tag{51}$$

$$\left.\begin{array}{l}\texttt{2ea\underline{b}}\\ \texttt{3ea\underline{b}}\end{array}\right\} \to \texttt{3\underline{e}bc} \to \texttt{2bab\_} \tag{52}$$

$$\texttt{1ea\underline{b}} \to \texttt{3\underline{e}bd} \to \texttt{2bdb\_} \tag{53}$$

$$\begin{array}{l}\texttt{3ae\underline{c}} \to \texttt{3\underline{a}ca} \to \texttt{4ccc\_}\\ \texttt{1ae\underline{a}} \to \texttt{3\underline{a}cd} \to \texttt{2cdb\_}\\ \texttt{5ae\underline{a}} \to \texttt{3\underline{a}ce} \to \texttt{3ccb\_}\end{array} \tag{54}$$

$$\left.\begin{array}{l}\texttt{3ab\underline{d}}\\ \texttt{4ab\underline{d}}\end{array}\right\} \to \texttt{3\underline{a}ec} \to \texttt{3caa\_} \tag{55}$$

$$\left.\begin{array}{l}\texttt{3eb\underline{d}}\\ \texttt{4eb\underline{d}}\end{array}\right\} \to \texttt{3\underline{e}ec} \to \texttt{4bcc\_} \tag{56}$$

$$\left.\begin{array}{l}\texttt{4\underline{e}cb}\\ \texttt{4\underline{e}eb}\end{array}\right\} \to \texttt{3aa\underline{b}} \to \texttt{3cbc\_} \tag{57}$$

$$\left.\begin{array}{l}\texttt{5\underline{c}ec}\\ \texttt{5\underline{e}ec}\end{array}\right\} \to \texttt{3ab\underline{c}} \to \texttt{3\_bca} \tag{58}$$

$$\texttt{3\underline{e}ec} \to \texttt{3bb\underline{c}} \to \texttt{4bcc\_} \tag{59}$$

$$\texttt{4\underline{c}ec} \to \texttt{3cb\underline{c}} \to \texttt{1abc\_} \tag{60}$$

$$\left.\begin{array}{l}\texttt{4\underline{e}cd}\\ \texttt{4\underline{e}ed}\end{array}\right\} \to \texttt{3aa\underline{d}} \to \texttt{1cbd\_} \tag{61}$$

$$\left.\begin{array}{l}\texttt{5\underline{c}ed}\\ \texttt{5\underline{e}ed}\end{array}\right\} \to \texttt{3ab\underline{d}} \to \texttt{3caa\_} \tag{62}$$

$$\begin{array}{l}\mathtt{3\underline{e}ed} \to \mathtt{3bb\underline{d}} \to \mathtt{4\_cec}\\ \mathtt{4\underline{c}ed} \to \mathtt{3cb\underline{d}} \to \mathtt{2\_aec}\end{array} \tag{63}$$

$$\left.\begin{array}{l}\mathtt{3}\alpha_2\mathtt{b\underline{c}}\\ \mathtt{1}\alpha_2\mathtt{b\underline{c}}\end{array}\right\} \to \mathtt{4}\underline{\alpha_2}\mathtt{ca}\begin{cases}\overset{\alpha_2=\mathtt{b}}{\to} \mathtt{4bcc\_}\\ \overset{\alpha_2=\mathtt{c}}{\to} \mathtt{1abc\_}\\ \overset{\alpha_2=\mathtt{e}}{\to} \mathtt{4aac\_}\end{cases} \tag{64}$$

$$\mathtt{4}\alpha_2\mathtt{b\underline{a}} \to \mathtt{4}\underline{\alpha_2}\mathtt{cb}\begin{cases}\overset{\alpha_2=\mathtt{b}}{\to} \mathtt{2bab\_}\\ \overset{\alpha_2=\mathtt{c}}{\to} \mathtt{3abc\_}\\ \overset{\alpha_2=\mathtt{e}}{\to} \mathtt{3cbc\_}\end{cases} \tag{65}$$

$$\mathtt{2}\alpha_2\mathtt{b\underline{e}} \to \mathtt{4}\underline{\alpha_2}\mathtt{cc}\begin{cases}\overset{\alpha_2=\mathtt{b}}{\to} \mathtt{2bab\_}\\ \overset{\alpha_2=\mathtt{c}}{\to} \mathtt{1abd\_}\\ \overset{\alpha_2=\mathtt{e}}{\to} \mathtt{2adb\_}\end{cases} \tag{66}$$

$$\mathtt{1}\alpha_2\mathtt{b\underline{a}} \to \mathtt{4}\underline{\alpha_2}\mathtt{cd}\begin{cases}\overset{\alpha_2=\mathtt{b}}{\to} \mathtt{2bdb\_}\\ \overset{\alpha_2=\mathtt{c}}{\to} \mathtt{1aba\_}\\ \overset{\alpha_2=\mathtt{e}}{\to} \mathtt{1cbd\_}\end{cases} \tag{67}$$

$$\left.\begin{array}{l}\mathtt{5}\alpha_2\mathtt{b\underline{a}}\\ \mathtt{5}\alpha_2\mathtt{b\underline{b}}\end{array}\right\} \to \mathtt{4}\underline{\alpha_2}\mathtt{ce}\begin{cases}\overset{\alpha_2=\mathtt{b}}{\to} \mathtt{3bcb\_}\\ \overset{\alpha_2=\mathtt{c}}{\to} \mathtt{3\_bcc}\\ \overset{\alpha_2=\mathtt{e}}{\to} \mathtt{3aab\_}\end{cases} \tag{68}$$

$$\left.\begin{array}{l}\mathtt{3cd\underline{d}}\\ \mathtt{4cd\underline{d}}\end{array}\right\} \to \mathtt{4\underline{c}ac} \to \mathtt{3ccc\_} \tag{69}$$

$$\left.\begin{array}{l}\mathtt{3ed\underline{d}}\\ \mathtt{4ed\underline{d}}\end{array}\right\} \to \mathtt{4\underline{e}ac} \to \mathtt{1cbd\_} \tag{70}$$

$$\mathtt{4\underline{b}b}\alpha_2 \to \mathtt{4bb}\underline{\alpha_2}\begin{cases}\overset{\alpha_2=\mathtt{a}}{\to} \mathtt{2bab\_}\\ \overset{\alpha_2=\mathtt{d}}{\to} \mathtt{4\_cec}\end{cases} \tag{71}$$

$$\mathtt{3\underline{a}b}\alpha_2 \to \mathtt{4cb}\underline{\alpha_2}\begin{cases}\overset{\alpha_2=\mathtt{a}}{\to} \mathtt{3abc\_}\\ \overset{\alpha_2=\mathtt{d}}{\to} \mathtt{2\_aec}\end{cases} \tag{72}$$

$$\left.\begin{array}{l}\mathtt{5\underline{c}aa}\\ \mathtt{5\underline{e}aa}\end{array}\right\} \to \mathtt{4ac\underline{a}} \to \mathtt{3dbc\_} \tag{73}$$

$$\mathtt{3\underline{e}aa} \to \mathtt{4bc\underline{a}} \to \mathtt{4\_cab} \tag{74}$$

$$\mathtt{4\underline{c}aa} \to \mathtt{4cc\underline{a}} \to \mathtt{3abc\_} \tag{75}$$

$$\left.\begin{array}{l}\texttt{1}\alpha_2\texttt{d}\underline{\texttt{c}}\\ \texttt{5}\alpha_2\texttt{d}\underline{\texttt{d}}\end{array}\right\} \rightarrow \texttt{5}\underline{\alpha_2}\texttt{ca} \rightarrow \texttt{1dbc\_} \text{ for } \alpha_2 \in \{\texttt{c},\texttt{e}\} \tag{76}$$

$$\texttt{4}\alpha_2\texttt{d}\underline{\texttt{a}} \rightarrow \texttt{5}\underline{\alpha_2}\texttt{cb} \rightarrow \texttt{3dbc\_} \text{ for } \alpha_2 \in \{\texttt{c},\texttt{e}\} \tag{77}$$

$$\left.\begin{array}{l}\texttt{2}\alpha_2\texttt{d}\underline{\texttt{b}}\\ \texttt{3}\alpha_2\texttt{d}\underline{\texttt{b}}\\ \texttt{2}\alpha_2\texttt{d}\underline{\texttt{e}}\end{array}\right\} \rightarrow \texttt{5}\underline{\alpha_2}\texttt{cc} \rightarrow \texttt{1dbd\_} \text{ for } \alpha_2 \in \{\texttt{c},\texttt{e}\} \tag{78}$$

$$\texttt{1}\alpha_2\texttt{d}\underline{\texttt{b}} \rightarrow \texttt{5}\underline{\alpha_2}\texttt{cd} \rightarrow \texttt{1dba\_} \text{ for } \alpha_2 \in \{\texttt{c},\texttt{e}\} \tag{79}$$

$$\texttt{5}\alpha_2\texttt{d}\underline{\texttt{b}} \rightarrow \texttt{5}\underline{\alpha_2}\texttt{ce} \rightarrow \texttt{5\_ccc} \text{ for } \alpha_2 \in \{\texttt{c},\texttt{e}\} \tag{80}$$

$$\begin{array}{l}\texttt{4}\underline{\texttt{b}}\texttt{ed} \rightarrow \texttt{5ba}\underline{\texttt{d}} \rightarrow \texttt{4\_cba}\\ \texttt{3}\underline{\texttt{a}}\texttt{ed} \rightarrow \texttt{5ca}\underline{\texttt{d}} \rightarrow \texttt{2\_aba}\end{array} \tag{81}$$

These are in agreement with the 78 IRR(3) found by the computer program [3].
The numbers of IRR found for the TM for lengths 2-10 are 55,78,163,291,702,1578,3958,9686, and 24631 respectively.

2 A systematic approach to generating the IRR's

Indicating CS's by just the the pointer positions in typewriter font, any rule of the form $\texttt{1} \rightarrow \texttt{n}$ of length n demonstrates the reachability of the CS $\texttt{n}$ regardless of whether or not there is another CS of the form $\texttt{n}$ between the CS's $\texttt{1}$ and $\texttt{n}$. If there is no other $\texttt{1}$ between them, the $\texttt{1}$ is an origin. A preceding CS of the form $\texttt{1}$ would not be recorded as an origin because the backward searching algorithm would stop when the first $\texttt{1}$ searching backwards is reached, and it is unnecessary to do so. This follows from the backward searching algorithm in section 2.2 of [2]. Therefore

Lemma 2.1. *A computation represented by* $\texttt{1} \rightarrow \texttt{n} \rightarrow \texttt{0}$ *or* $\texttt{n} \rightarrow \texttt{1} \rightarrow \texttt{n}+\texttt{1}$ *represents a triplet for an extendable IRR (type RL or LR respectively) if and only if the origin indicated is the first one arrived at after tracing the computation back from the LHS i.e there is no other CS* $\texttt{1}$ *in the first case, and no other CS* $\texttt{n}$ *in the second case between the* $\texttt{1}$ *and the* $\texttt{n}$.

Generating the IRR starts with all single TM steps in the above notation $\texttt{1} \rightarrow \texttt{0}$ (i.e. $\underline{\texttt{x}} \rightarrow \texttt{\_x}$) or $\texttt{1} \rightarrow \texttt{2}$ (i.e. $\underline{\texttt{x}} \rightarrow \texttt{x\_}$) where $\texttt{x}$'s represents an arbitrary symbol that could be different for each use. In the first case, adding the arbitrary symbol α on the left and continuing the computation as far as possible gives results either of the form (i) $\texttt{2} \rightarrow \texttt{1} \rightarrow \texttt{3}$ or (ii) $\texttt{2} \rightarrow \texttt{1} \rightarrow \texttt{0}$ i.e. $\alpha\underline{\texttt{x}} \rightarrow$

$\underline{\alpha}\mathtt{x} \to \mathtt{xx\_}$ or $\mathtt{\_xx}$ respectively. The results in case (i) are IRR by Lemma 2.1 because there can be no other CS's between the $\mathtt{2}$ and the $\mathtt{1}$. The results from (ii) are IRR's if and only if the first move beyond the $\mathtt{1}$ is to $\mathtt{2}$ i.e. the computation has the form $\mathtt{2} \to \mathtt{1} \to \mathtt{2} \to \mathtt{0}$. This ensures that the rule $\mathtt{1} \to \mathtt{0}$ contained in (ii) of length 2 is irreducible. Likewise for the mirror image case starting with a rule of the form $\mathtt{1} \to \mathtt{2}$ adding the α on the right and continuing gives results of the form $\mathtt{1} \to \mathtt{2} \to \mathtt{0}$ ($\in$ IRR($\mathtt{2}$)) or of the form $\mathtt{1} \to \mathtt{2} \to \mathtt{3}$ ($\in$ IRR($\mathtt{2}$) if and only if the first move from the $\mathtt{2}$ is to $\mathtt{1}$).

Continuing from this, let $\mathtt{X}$ be a member of IRR($\mathtt{n}$). Then $\mathtt{X}$ can be represented as $\mathtt{A} \to \mathtt{B} \to \mathtt{C}$ where the pointer swaps ends between $\mathtt{A}$ and $\mathtt{B}$ to establish the reachability of $\mathtt{B}$ necessary for $\mathtt{X}$ to be an IRR. There may be more than one such CS $\mathtt{A}$ for a single $\mathtt{B}$ and the set of all such $\mathtt{A}$ will be denoted by $\mathtt{O_1}(\mathtt{B})$ (the same as $\mathtt{S}(\mathtt{B})$ in [2]), the 1 referring to the backward searching algorithm that terminates in condition 1. (Likewise if it terminates in condition 2, the set of such endpoints will be denoted as $\mathtt{O_2}(\mathtt{B})$, but these do not confer reachability and will not be referred to as origins.) If the pointer is at the right in $\mathtt{A}$ and at the left in $\mathtt{B}$ then at $\mathtt{C}$ it can be at the left or right so that $\mathtt{X}$ must be represented as either of the triplet forms $\mathtt{n} \to \mathtt{1} \to \mathtt{0}$ or $\mathtt{n} \to \mathtt{1} \to \mathtt{n+1}$ respectively and having types LL and LR respectively. Likewise if the pointer is at the left in $\mathtt{A}$ (the mirror image forms), $\mathtt{X}$ must be represented by either of $\mathtt{1} \to \mathtt{n} \to \mathtt{n+1}$ or $\mathtt{1} \to \mathtt{n} \to \mathtt{0}$, having types RR and RL respectively.

Either the backward search from $\mathtt{A}$ or the forward computation from $\mathtt{C}$ (or both) may lead to a CS that has arisen before. These are stationary cycling cases. If it happens in the backward search, the effect is the same as when condition 3 occurs i.e. when no further backward steps are possible and this branch of the search ends with no new origins. In the forward computation from $\mathtt{C}$ this prevents a genuine RHS to be found. The IRR is then of type $\mathtt{LC}$ or $\mathtt{RC}$. No new IRR's can be derived from these cases.

In order to generate all the IRR $\mathtt{Y}$ of length n+1 based on the single IRR $\mathtt{X}$ of length n and having the form $\mathtt{n} \to \mathtt{1} \to \mathtt{n+1}$, consider first $\mathtt{A}\alpha \to \mathtt{B}\alpha \to \mathtt{C}\alpha$ where α is any symbol the TM uses. Clearly by Theorem 9.1 of [1] every such IRR $\mathtt{Y}$ can be obtained starting from the LHS $\mathtt{B}\alpha$ if an appropriate α can be found. The symbol α must be chosen so that $\mathtt{B}\alpha$ is reachable and the origins $\mathtt{O_1}(\mathtt{B}\alpha) \neq \emptyset$ for it need to be found. These are all the terminal CS's of length $\mathtt{n+1}$ from the backward searching algorithm starting from $\mathtt{B}\alpha$ and ending in condition (1). Each of these branches has a point where the pointer first reaches $\mathtt{n}$ and this set of CS's is $\mathtt{A}\alpha$ because the α has yet played no part, so $\mathtt{O_1}(\mathtt{B}\alpha)$ is the union of the backward searches $\mathtt{O_1}$ applied to all the CS's in $\mathtt{A}\alpha$, thus the backward search algorithm is applied to $\mathtt{A}\alpha$ and identifying all possible values of α i.e. the values of α for which $\mathtt{O_1}(\mathtt{A}\alpha) \neq \emptyset$ and generating all its origins for each such α. Also the forward computation from $\mathtt{C}\alpha$ is continued as far as possible to generate the RHS of $\mathtt{Y}$ and hence what its type is (RR, RL, LR, LL, RC, or LC) the last two cases coming from a stationary cycle in the forward computation from $\mathtt{C}\alpha$. If the pointer is at the left in $\mathtt{A}$ and $\mathtt{C}$ and the right in $\mathtt{B}$ (the mirror image case) the added arbitrary symbol α will be on the left. This procedure for generating the all the IRR $\mathtt{Y}$ of length $\mathtt{n}+\mathtt{1}$ like this from an IRR $\mathtt{X}$ of length $\mathtt{n}$, including the mirror image case where the triplet form of $\mathtt{X}$ is $\mathtt{1} \to \mathtt{n} \to \mathtt{0}$, will denoted by the function $\mathtt{F}$. This proves that

Theorem 2.2. *Every member of IRR($\mathtt{n}+\mathtt{1}$) can be obtained using* $\mathtt{F}$ *from some* $\mathtt{X} \in$ *IRR*($\mathtt{n}$). *Also, because an IRR is uniquely determined by is LHS, the sets of IRR obtained like this for different* $\mathtt{X}$ *(different* $\mathtt{B}$*) must be disjoint i.e.* $\mathtt{F}^{-1}$ *applied to a member of IRR($\mathtt{n}+\mathtt{1}$) is a unique member of IRR($\mathtt{n}$). The first part can be written symbolically as*

$$IRR(\mathtt{n}+\mathtt{1}) = \bigcup_{\mathtt{X}\in\mathtt{IRR(n)}} \{\mathtt{F}(\mathtt{X})\} \tag{82}$$

To illustrate this, the following is an outline showing an IRR of length n of type RLR (type LR with origin having the pointer at the right) as above and the possible types of result (except the cases where a stationary cycle occurs) of this argument for a given symbol α that could include a new IRR triplet of length $\mathtt{n}+\mathtt{1}$.

$$\left.\begin{array}{r} \left.\begin{array}{r}\text{Cannot be used to}\\ \text{prove reachability}\end{array}\right\} \\ \text{Does prove reachability}\end{array}\begin{array}{cc} \mathtt{1} & \to \\ \mathtt{n}+\mathtt{1} & \to \end{array}\right\} \mathtt{n} \to \mathtt{1} \to \mathtt{n{+}1} \left\{\begin{array}{ccl} \to & \mathtt{0} & \text{type LL} \\ \to & \mathtt{n}+\mathtt{2} & \text{type LR}\end{array}\right. \tag{83}$$

The 3 central CS's refer to a member of IRR($\mathtt{n}$), and the leftmost, central, and rightmost CS's refer to the corresponding member of IRR($\mathtt{n}+\mathtt{1}$) if reachability of the CS $\mathtt{1}$ is found. The corresponding mirror image result for the LRL case is as follows:

$$\left.\begin{array}{r} \text{Does prove reachability} \\ \left.\begin{array}{r}\text{Cannot be used to}\\ \text{prove reachability}\end{array}\right\}\end{array}\begin{array}{cc} \mathtt{1} & \to \\ \mathtt{n}+\mathtt{1} & \to \end{array}\right\} \mathtt{2} \to \mathtt{n{+}1} \to \mathtt{1} \left\{\begin{array}{ccl} \to & \mathtt{0} & \text{type RL} \\ \to & \mathtt{n}+\mathtt{2} & \text{type RR}\end{array}\right. \tag{84}$$

In this case note that because the α is added on the left, all the pointer positions in the IRR of length $\mathtt{n}$ have been increased by 1 when they appear in the embedded IRR of length $\mathtt{n}$, so originally they would be $\mathtt{1} \to \mathtt{n} \to \mathtt{0}$.

The above procedure allows the generation of all the IRR of a TM up to any given length and has been implemented [3]. In the remainder of this paper, this procedure will be adapted to the incomplete description of IRR's (IRR outlines with ...) needed to make the analysis manageable for the example TM, that is expected to have an infinite number of IRR's.

3 Using abbreviated IRR's

Abbreviated IRR's have been found to be very useful because they result in shortening of the lists of cases to be considered, and sets of IRR's that match the abbreviated IRR's often have similar derivations of new IRR's one symbol longer, which can be dealt with together. The latter is because in (83) the derivation involves two derivations one of each of the forms $\mathtt{n}+\mathtt{1}\to\mathtt{n}$ and $\mathtt{n}+\mathtt{1}\to\mathtt{n}+\mathtt{2}$ and it is to be expected that these will often involve one or a small number of TM steps. A similar remark applies for the case (84) too. Also the treatment of IRR's in TM1 in [2] where sets conforming to the patterns or outline IRR's could be treated in the same way, allowed a proof of a general method for obtaining all the IRR($\mathtt{n}+\mathtt{1}$) from the IRR($\mathtt{n}$) for all sufficiently large $\mathtt{n}$ for this TM.

The abbreviations done here are (1) to a fixed length $\mathtt{k}$ of the strings of symbols, where the symbol at the pointer in the origin is at one end and (2) the LHS is removed completely. The abbreviation is indicated by $\ldots$ in place of the deleted symbols in the origin and the RHS. This presentation has the property that the pointer position appears in the RHS column if and only if the IRR is of extendable type i.e. type RL or LR. Table 1 was obtained by abbreviating all the IRR(3) to length $\mathtt{k}=\mathtt{2}$. For this purpose, IRR's with many origins have a separate entry for each origin. This allows the origins to be unique for each row in the tables.

Table 1: List of patterns for the IRR(3)

Origin	RHS's	Origin	RHS's
$\mathtt{1}\underline{\mathtt{d}}\mathtt{d}\ldots$	$\mathtt{4\_cb}\ldots$	$\mathtt{5}\underline{\mathtt{e}}\mathtt{e}\ldots$	$\{\mathtt{3\_bc}\ldots,\mathtt{3ca}\ldots\}$
$\mathtt{1}\underline{\mathtt{e}}\mathtt{d}\ldots$	$\mathtt{4\_cb}\ldots$	$\mathtt{3}\underline{\mathtt{e}}\mathtt{e}\ldots$	$\{\mathtt{4bc}\ldots,\mathtt{4\_ce}\ldots\}$
$\mathtt{1}\underline{\mathtt{d}}\mathtt{d}\ldots$	$\mathtt{4\_ca}\ldots$	$\mathtt{4}\underline{\mathtt{e}}\mathtt{c}\ldots$	$\{\mathtt{3cb}\ldots,\mathtt{1cb}\ldots\}$
$\mathtt{1}\underline{\mathtt{e}}\mathtt{d}\ldots$	$\mathtt{4\_ca}\ldots$	$\mathtt{4}\underline{\mathtt{e}}\mathtt{e}\ldots$	$\{\mathtt{3cb}\ldots,\mathtt{1cb}\ldots\}$
$\mathtt{1}\underline{\mathtt{d}}\mathtt{a}\ldots$	$\mathtt{4\_ca}\ldots$	$\mathtt{4}\underline{\mathtt{c}}\mathtt{e}\ldots$	$\{\mathtt{1ab}\ldots,\mathtt{2\_ae}\ldots\}$
$\mathtt{1}\underline{\mathtt{e}}\mathtt{a}\ldots$	$\mathtt{4\_ca}\ldots$	$\mathtt{3}\ldots\mathtt{b}\underline{\mathtt{c}}$	$\{\mathtt{4}\ldots\mathtt{cc\_},\mathtt{1}\ldots\mathtt{bc\_},\mathtt{4}\ldots\mathtt{ac\_}\}$
$\mathtt{1}\underline{\mathtt{d}}\mathtt{c}\ldots$	$\mathtt{3\_ec}\ldots$	$\mathtt{1}\ldots\mathtt{b}\underline{\mathtt{c}}$	$\{\mathtt{4}\ldots\mathtt{cc\_},\mathtt{1}\ldots\mathtt{bc\_},\mathtt{4}\ldots\mathtt{ac\_}\}$
$\mathtt{1}\underline{\mathtt{e}}\mathtt{c}\ldots$	$\mathtt{3\_ec}\ldots$	$\mathtt{4}\ldots\mathtt{b}\underline{\mathtt{a}}$	$\{\mathtt{2}\ldots\mathtt{ab\_},\mathtt{3}\ldots\mathtt{bc\_}\}$
$\mathtt{1}\ldots\mathtt{c}\underline{\mathtt{c}}$	$\mathtt{1}\ldots\mathtt{bc\_}$	$\mathtt{2}\ldots\mathtt{b}\underline{\mathtt{e}}$	$\{\mathtt{2}\ldots\mathtt{ab\_},\mathtt{1}\ldots\mathtt{bd\_},\mathtt{2}\ldots\mathtt{db\_}\}$
$\mathtt{4}\ldots\mathtt{c}\underline{\mathtt{a}}$	$\mathtt{3}\ldots\mathtt{bc\_}$	$\mathtt{1}\ldots\mathtt{b}\underline{\mathtt{a}}$	$\{\mathtt{2}\ldots\mathtt{db\_},\mathtt{1}\ldots\mathtt{ba\_},\mathtt{1}\ldots\mathtt{bd\_}\}$
$\mathtt{2}\ldots\mathtt{c}\underline{\mathtt{e}}$	$\mathtt{1}\ldots\mathtt{bd\_}$	$\mathtt{5}\ldots\mathtt{b}\underline{\mathtt{a}}$	$\{\mathtt{3}\ldots\mathtt{cb\_},\mathtt{3}\ldots\mathtt{cc},\mathtt{3}\ldots\mathtt{ab\_}\}$
$\mathtt{5}\ldots\mathtt{c}\underline{\mathtt{b}}$	$\{\mathtt{5}\ldots\mathtt{cc},\mathtt{3}\ldots\mathtt{cc},\mathtt{1}\ldots\mathtt{bd\_}\}$	$\mathtt{5}\ldots\mathtt{b}\underline{\mathtt{b}}$	$\{\mathtt{3}\ldots\mathtt{cb\_},\mathtt{3}\ldots\mathtt{cc},\mathtt{3}\ldots\mathtt{ab\_}\}$
$\mathtt{3}\ldots\mathtt{a}\underline{\mathtt{d}}$	$\mathtt{1}\ldots\mathtt{bd\_}$	$\mathtt{4}\underline{\mathtt{b}}\mathtt{b}\ldots$	$\{\mathtt{2ba}\ldots,\mathtt{4\_ce}\ldots\}$
$\mathtt{4}\ldots\mathtt{a}\underline{\mathtt{d}}$	$\mathtt{1}\ldots\mathtt{bd\_}$	$\mathtt{3}\underline{\mathtt{a}}\mathtt{b}\ldots$	$\{\mathtt{3ab}\ldots,\mathtt{2\_ae}\ldots\}$
$\mathtt{2}\underline{\mathtt{d}}\mathtt{d}\ldots$	$\mathtt{3\_bc}\ldots$	$\mathtt{5}\underline{\mathtt{c}}\mathtt{a}\ldots$	$\mathtt{3db}\ldots$
$\mathtt{2}\underline{\mathtt{d}}\mathtt{e}\ldots$	$\mathtt{3\_bc}\ldots$	$\mathtt{5}\underline{\mathtt{e}}\mathtt{a}\ldots$	$\mathtt{3db}\ldots$
$\mathtt{2}\underline{\mathtt{a}}\mathtt{d}\ldots$	$\mathtt{1ab}\ldots$	$\mathtt{3}\underline{\mathtt{e}}\mathtt{a}\ldots$	$\mathtt{4\_ca}\ldots$

$\mathtt{2\underline{a}e\ldots}$	$\mathtt{1ab\ldots}$	$\mathtt{4\underline{c}a\ldots}$	$\mathtt{3ab\ldots}$
$\mathtt{2\underline{c}d\ldots}$	$\mathtt{5\_cc\ldots}$	$\mathtt{1\ldots d\underline{c}}$	$\mathtt{1\ldots bc\_}$
$\mathtt{2\underline{c}e\ldots}$	$\mathtt{5\_cc\ldots}$	$\mathtt{4\ldots d\underline{a}}$	$\mathtt{3\ldots bc\_}$
$\mathtt{5\ldots a\underline{d}}$	$\{\mathtt{3\ldots bc\_}, \mathtt{4\ldots cc\_}\}$	$\mathtt{2\ldots d\underline{e}}$	$\mathtt{1\ldots bd\_}$
$\mathtt{2\ldots a\underline{b}}$	$\{\mathtt{3\ldots bc\_}, \mathtt{2\ldots ab\_}\}$	$\mathtt{5\ldots d\underline{b}}$	$\mathtt{5\ldots cc}$
$\mathtt{3\ldots a\underline{b}}$	$\{\mathtt{3\ldots bc\_}, \mathtt{2\ldots ab\_}\}$	$\mathtt{5\ldots d\underline{d}}$	$\mathtt{1\ldots bc\_}$
$\mathtt{1\ldots a\underline{b}}$	$\{\mathtt{2\ldots ec}, \mathtt{2\ldots db\_}\}$	$\mathtt{2\ldots d\underline{b}}$	$\mathtt{1\ldots bd\_}$
$\mathtt{3\ldots e\underline{c}}$	$\mathtt{4\ldots cc\_}$	$\mathtt{3\ldots d\underline{b}}$	$\mathtt{1\ldots bd\_}$
$\mathtt{1\ldots e\underline{a}}$	$\mathtt{2\ldots db\_}$	$\mathtt{1\ldots d\underline{b}}$	$\mathtt{1\ldots ba\_}$
$\mathtt{5\ldots e\underline{a}}$	$\mathtt{3\ldots cb\_}$	$\mathtt{4\underline{b}e\ldots}$	$\mathtt{4\_cb\ldots}$
$\mathtt{3\ldots b\underline{d}}$	$\{\mathtt{3\ldots aa\_}, \mathtt{4\ldots cc\_}\}$	$\mathtt{3\underline{a}e\ldots}$	$\mathtt{2\_ab\ldots}$
$\mathtt{4\ldots b\underline{d}}$	$\{\mathtt{3\ldots aa\_}, \mathtt{4\ldots cc\_}\}$	$\mathtt{3\ldots d\underline{d}}$	$\{\mathtt{3\ldots cc\_}, \mathtt{1\ldots bd\_}\}$
$\mathtt{5\underline{c}e\ldots}$	$\{\mathtt{3\_bc\ldots}, \mathtt{3ca\ldots}\}$	$\mathtt{4\ldots d\underline{d}}$	$\{\mathtt{3\ldots cc\_}, \mathtt{1\ldots bd\_}\}$

Treating Table 1 at first as a hypothesis to be proved by induction for IRR of length $\mathtt{n} \geq \mathtt{3}$ where $\ldots$ represents any string of symbols of length $\geq \mathtt{1}$ that could be different whenever it is used, suggests (1) following the method in Section 2 to each of the patterns provided they are extendable i.e. of type LR or RL in Table 1, and then (2) truncating the result to two symbols from the pointer at the new origin so generating another result of the same form as those already in Table 1. This must repeated as necessary to ensure closure, i.e. every outline IRR($\mathtt{n}+\mathtt{1}$) must also appear as an outline IRR($\mathtt{n}$) resulting in Table 2. Thus an infinite number of IRR could be captured by the resulting recursive description. This procedure was done in order that every application of $\mathtt{F}$ (that always maps IRR's to IRR's and by repeated application generates all IRR's for a TM from the set IRR(3)) maps an IRR outline in Table 2 to another IRR outline in Table 2 unless the backward search procedure to find new origins goes in the unexpected direction. The analysis for these cases where the derivation of the new origin takes the pointer away from the α will be taken as far a possible consistent with the constraints of the other calculations and are therefore represented differently i.e. the new origin is given with the pointer opposite the α and not truncated to $\mathtt{k}=\mathtt{2}$ symbols (they have $\mathtt{k}=\mathtt{3}$), and α can take any of the 5 values. This issue is taken up again in Section 10.

For example starting from the fifth result in Table 1

$$\mathtt{1\underline{d}a\ldots} \rightarrow\rightarrow \mathtt{4\_ca\ldots} \tag{85}$$

(using (2).2) this gives

$$\begin{array}{llllll}
\text{deriving the origin} & & & \text{RHS} & \alpha \\
\alpha\mathtt{A} & & \alpha\mathtt{C} & & \\
\multirow{3}{*}{$\mathtt{1\alpha\underline{d}a\ldots}\left\{\begin{array}{l}\\ \\ \end{array}\right.$} & \overset{\alpha=\mathtt{a}}{\leftarrow} \mathtt{2\underline{d}da\ldots} & \mathtt{4\underline{a}ca\ldots} & \mathtt{3_bca\ldots} & \mathtt{a} \\
 & \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{2\underline{a}da\ldots} & \mathtt{4\underline{c}ca\ldots} & \mathtt{1abc_\ldots} & \mathtt{c} \\
 & \overset{\alpha=\mathtt{d}}{\leftarrow} \mathtt{2\underline{c}da\ldots} & \mathtt{4\underline{d}ca\ldots} & \mathtt{5_cca\ldots} & \mathtt{d}
\end{array} \tag{86}$$

which is truncated to

$$
\begin{aligned}
&\mathtt{2\underline{d}d}\ldots \rightarrow\rightarrow \mathtt{3\_bc}\ldots\\
&\mathtt{2\underline{a}d}\ldots \rightarrow\rightarrow \mathtt{1ab}\ldots\\
&\mathtt{2\underline{c}d}\ldots \rightarrow\rightarrow \mathtt{5\_cc}\ldots
\end{aligned}
\tag{87}
$$

for $\alpha = \mathtt{a}, \mathtt{c}, \mathtt{d}$ respectively. Note that in this argument, adding the arbitrary symbol α on the left (because the pointer in the origin is on the left) maintains the pointer being on the right hand end of the string of symbols in the LHS (not shown), and this is necessary to obtain an abbreviated IRR. The result of this argument is that if an IRR of length $\mathtt{n}$ conforms to (85), then there are 3 more IRR's of length $\mathtt{n}+\mathtt{1}$ corresponding to (87) for $\alpha \in \{\mathtt{a}, \mathtt{c}, \mathtt{d}\}$ respectively. The second of these results in (87) has the pointer in the RHS on the right (not shown due to truncation), so this IRR has type RR and cannot be used to derive other IRR.

These are examples of rules that generate IRR's of length $\mathtt{n}+\mathtt{1}$ from other IRR's of length $\mathtt{n}$ and are expressed (for the case where the unknown symbols are on the right) in Table 2 for $\mathtt{k} = 2$. Collectively such rules will be referred to as IRR-generating rules (IGR) to distinguish them from computation rules.

The longest possible tape length to derive a triplet of the form $\mathtt{n} \rightarrow \mathtt{1} \rightarrow \mathtt{n}+\mathtt{1}$ in the notation of (83) from one of the form $\mathtt{n}-\mathtt{1} \rightarrow \mathtt{1} \rightarrow \mathtt{n}$ would involve the pointer moving in the backward search thus: $\mathtt{n}-\mathtt{1} \leftarrow \mathtt{2} \leftarrow \mathtt{n}$ and would in general be $\mathtt{n}-\mathtt{1} \leftarrow \mathtt{n}-\mathtt{k} \leftarrow \mathtt{n}$. It could not reach point $\mathtt{1}$ because that would terminate the backward search algorithm. Therefore $2 \leq \mathtt{k} \leq \mathtt{n}-2$. It is interesting to note that the backward search in general takes the form of an IRR triplet of length $\mathtt{k}-\mathtt{1}$ run in reverse.

Table 2: Outline IRRs of length $\mathtt{n}$ and their derived outline IRR of length $\mathtt{n}+\mathtt{1}$, with the unknown symbols on the right.

set number	IRR($\mathtt{n}$) Origin	RHS	α	IRR($\mathtt{n}+\mathtt{1}$) Origin	RHS
1	$\mathtt{1\underline{d}a}\ldots$	$\mathtt{4\_ca}\ldots$	a	$\mathtt{2\underline{d}d}\ldots$	$\mathtt{3\_bc}\ldots$
			c	$\mathtt{2\underline{a}d}\ldots$	$\mathtt{1ab}\ldots$
			d	$\mathtt{2\underline{c}d}\ldots$	$\mathtt{5\_cc}\ldots$
2	$\mathtt{1\underline{d}c}\ldots$	$\mathtt{3\_ec}\ldots$	a	$\mathtt{2\underline{d}d}\ldots$	$\mathtt{3ca}\ldots$
			c	$\mathtt{2\underline{a}d}\ldots$	$\mathtt{2\_ae}\ldots$
			d	$\mathtt{2\underline{c}d}\ldots$	$\mathtt{5\_ce}\ldots$
3	$\mathtt{1\underline{d}d}\ldots$	$\mathtt{4\_ca}\ldots$	a	$\mathtt{2\underline{d}d}\ldots$	$\mathtt{3\_bc}\ldots$
			c	$\mathtt{2\underline{a}d}\ldots$	$\mathtt{1ab}\ldots$
			d	$\mathtt{2\underline{c}d}\ldots$	$\mathtt{5\_cc}\ldots$
4	$\mathtt{1\underline{d}d}\ldots$	$\mathtt{4\_cb}\ldots$	a	$\mathtt{2\underline{d}d}\ldots$	$\mathtt{3\_bc}\ldots$
			c	$\mathtt{2\underline{a}d}\ldots$	$\mathtt{3ab}\ldots$
			d	$\mathtt{2\underline{c}d}\ldots$	$\mathtt{5\_cc}\ldots$
5	$\mathtt{1\underline{e}a}\ldots$	$\mathtt{4\_ca}\ldots$	a	$\mathtt{2\underline{d}e}\ldots$	$\mathtt{3\_bc}\ldots$
			c	$\mathtt{2\underline{a}e}\ldots$	$\mathtt{1ab}\ldots$
			d	$\mathtt{2\underline{c}e}\ldots$	$\mathtt{5\_cc}\ldots$
6	$\mathtt{1\underline{e}c}\ldots$	$\mathtt{3\_ec}\ldots$	a	$\mathtt{2\underline{d}e}\ldots$	$\mathtt{3ca}\ldots$
			c	$\mathtt{2\underline{a}e}\ldots$	$\mathtt{2\_ae}\ldots$
			d	$\mathtt{2\underline{c}e}\ldots$	$\mathtt{5\_ce}\ldots$
7	$\mathtt{1\underline{e}d}\ldots$	$\mathtt{4\_ca}\ldots$	a	$\mathtt{2\underline{d}e}\ldots$	$\mathtt{3\_bc}\ldots$

		c	2$\underline{a}$e... 1ab...
		d	2$\underline{c}$e... 5_cc...
8	1$\underline{e}$d... 4_cb...	a	2$\underline{d}$e... 3_bc...
		c	2$\underline{a}$e... 3ab...
		d	2$\underline{c}$e... 5_cc...
9	2$\underline{a}$d... 2_ae...	b	1$\underline{d}$a... 4_ca...
		b	1$\underline{e}$a... 4_ca...
		α	1αa$\underline{a}$... RHS
10	2$\underline{a}$d... 1ab...	$\emptyset$	$\emptyset$
11	2$\underline{a}$d... 3ab...	$\emptyset$	$\emptyset$
12	2$\underline{a}$e... 2_ae...	b	1$\underline{d}$a... 4_ca...
		b	1$\underline{e}$a... 4_ca...
		α	5αa$\underline{a}$... RHS
13	2$\underline{a}$e... 1ab...	$\emptyset$	$\emptyset$
14	2$\underline{a}$e... 3ab...	$\emptyset$	$\emptyset$
15	2$\underline{c}$d... 5_cc...	b	1$\underline{d}$c... 3_ec...
		b	1$\underline{e}$c... 3_ec...
		α	1αc$\underline{a}$... RHS
16	2$\underline{c}$d... 5_ce...	b	1$\underline{d}$c... 3_ec...
		b	1$\underline{e}$c... 3_ec...
		α	1αc$\underline{a}$... RHS
17	2$\underline{c}$e... 5_cc...	b	1$\underline{d}$c... 3_ec...
		b	1$\underline{e}$c... 3_ec...
		α	5αc$\underline{a}$... RHS
18	2$\underline{c}$e... 5_ce...	b	1$\underline{d}$c... 3_ec...
		b	1$\underline{e}$c... 3_ec...
		α	5αc$\underline{a}$... RHS
19	2$\underline{d}$d... 3_bc...	b	1$\underline{d}$d... 4_cb...
		b	1$\underline{e}$d... 4_cb...
		α	1αd$\underline{a}$... RHS
20	2$\underline{d}$d... 3ca...	$\emptyset$	$\emptyset$
21	2$\underline{d}$e... 3_bc...	b	1$\underline{d}$d... 4_cb...
		b	1$\underline{e}$d... 4_cb...
		α	5αd$\underline{a}$... RHS
22	2$\underline{d}$e... 3ca...	$\emptyset$	$\emptyset$
23	3$\underline{a}$b... 2_ae...	a	5$\underline{c}$a... 5_cc...
		a	5$\underline{e}$a... 5_cc...
		b	3$\underline{e}$a... 4_ca...
		c	4$\underline{c}$a... 3_bc...
		α	4αa$\underline{a}$... RHS
24	3$\underline{a}$b... 3_bc...	a	5$\underline{c}$a... 3cb...
		a	5$\underline{e}$a... 3cb...
		b	3$\underline{e}$a... 4_cb...
		c	4$\underline{c}$a... 2_ab...
		α	4αa$\underline{a}$... RHS
25	3$\underline{a}$b... 1ab...	$\emptyset$	$\emptyset$
26	3$\underline{a}$b... 3ab...	$\emptyset$	$\emptyset$
27	3$\underline{a}$c... 2_ab...	a	5$\underline{c}$a... 3db...
		a	5$\underline{e}$a... 3db...
		b	3$\underline{e}$a... 4_ca...
		c	4$\underline{c}$a... 3ab...
		α	2αa$\underline{e}$... RHS
28	3$\underline{a}$c... 3_bc...	a	5$\underline{c}$a... 3cb...
		a	5$\underline{e}$a... 3cb...
		b	3$\underline{e}$a... 4_cb...
		c	4$\underline{c}$a... 2_ab...
		α	2αa$\underline{e}$... RHS
29	3$\underline{a}$c... 3ab...	$\emptyset$	$\emptyset$
30	3$\underline{a}$e... 2_ab...	a	5$\underline{c}$a... 3db...

		a	5$\underline{\text{e}}$a...	3db...
		b	3$\underline{\text{e}}$a...	4_ca...
		c	4$\underline{\text{c}}$a...	3ab...
		α	5αa$\underline{\text{b}}$...	RHS
31	3$\underline{\text{a}}$e... 1db...	$\emptyset$	$\emptyset$	
32	3$\underline{\text{e}}$a... 4_ca...	a	5$\underline{\text{c}}$e...	3_bc...
		a	5$\underline{\text{e}}$e...	3_bc...
		b	3$\underline{\text{e}}$e...	4bc...
		c	4$\underline{\text{c}}$e...	1ab...
		α	1αe$\underline{\text{c}}$...	RHS
33	3$\underline{\text{e}}$a... 4_cb...	a	5$\underline{\text{c}}$e...	3_bc...
		a	5$\underline{\text{e}}$e...	3_bc...
		b	3$\underline{\text{e}}$e...	2ba...
		c	4$\underline{\text{c}}$e...	3ab...
		α	1αe$\underline{\text{c}}$...	RHS
34	3$\underline{\text{e}}$e... 4_ce...	a	5$\underline{\text{c}}$e...	3_bc...
		a	5$\underline{\text{e}}$e...	3_bc...
		b	3$\underline{\text{e}}$e...	3bc...
		c	4$\underline{\text{c}}$e...	3_bc...
		α	5αe$\underline{\text{b}}$...	RHS
35	3$\underline{\text{e}}$e... 2ba...	$\emptyset$	$\emptyset$	
36	3$\underline{\text{e}}$e... 3bc...	$\emptyset$	$\emptyset$	
37	3$\underline{\text{e}}$e... 4bc...	$\emptyset$	$\emptyset$	
38	4$\underline{\text{b}}$b... 4_ce...	b	4$\underline{\text{b}}$b...	3bc...
		c	3$\underline{\text{a}}$b...	3_bc...
39	4$\underline{\text{b}}$b... 2ba...	$\emptyset$	$\emptyset$	
40	4$\underline{\text{b}}$b... 3bc...	$\emptyset$	$\emptyset$	
41	4$\underline{\text{b}}$b... 4bc...	$\emptyset$	$\emptyset$	
42	4$\underline{\text{b}}$c... 4_ca...	b	4$\underline{\text{b}}$b...	4bc...
		c	3$\underline{\text{a}}$b...	1ab...
		α	2αb$\underline{\text{b}}$...	RHS
		α	3αb$\underline{\text{b}}$...	RHS
43	4$\underline{\text{b}}$c... 4_cb...	b	4$\underline{\text{b}}$b...	2ba...
		c	3$\underline{\text{a}}$b...	3ab...
		α	2αb$\underline{\text{b}}$...	RHS
		α	3αb$\underline{\text{b}}$...	RHS
44	4$\underline{\text{b}}$e... 4_cb...	b	4$\underline{\text{b}}$b...	2ba...
		c	3$\underline{\text{a}}$b...	3ab...
45	4$\underline{\text{b}}$e... 4_ce...	b	4$\underline{\text{b}}$b...	3bc...
		c	3$\underline{\text{a}}$b...	3_bc...
46	4$\underline{\text{c}}$a... 2_ab...	b	4$\underline{\text{b}}$c...	4_ca...
		c	3$\underline{\text{a}}$c...	3ab...
		α	5αc$\underline{\text{d}}$...	RHS
47	4$\underline{\text{c}}$a... 3_bc...	b	4$\underline{\text{b}}$c...	4_cb...
		c	3$\underline{\text{a}}$c...	2_ab...
		α	5αc$\underline{\text{d}}$...	RHS
48	4$\underline{\text{c}}$a... 3ab...	$\emptyset$	$\emptyset$	
49	4$\underline{\text{c}}$e... 2_ae...	b	4$\underline{\text{b}}$c...	4_ca...
		c	3$\underline{\text{a}}$c...	3_bc...
50	4$\underline{\text{c}}$e... 3_bc...	b	4$\underline{\text{b}}$c...	4_cb...
		c	3$\underline{\text{a}}$c...	2_ab...
51	4$\underline{\text{c}}$e... 1ab...	$\emptyset$	$\emptyset$	
52	4$\underline{\text{c}}$e... 3ab...	$\emptyset$	$\emptyset$	
53	4$\underline{\text{e}}$c... 2_ec...	b	4$\underline{\text{b}}$e...	4_ce...
		c	3$\underline{\text{a}}$e...	1db...
		α	2αe$\underline{\text{b}}$...	RHS
		α	3αe$\underline{\text{b}}$...	RHS
54	4$\underline{\text{e}}$c... 1cb...	$\emptyset$	$\emptyset$	
55	4$\underline{\text{e}}$c... 3cb...	$\emptyset$	$\emptyset$	

56	$\mathtt{4\underline{e}e\ldots}$ $\mathtt{2\_ec\ldots}$	b c	$\mathtt{4\underline{b}e\ldots}$ $\mathtt{3\underline{a}e\ldots}$	$\mathtt{4\_ce\ldots}$ $\mathtt{1db\ldots}$
57	$\mathtt{4\underline{e}e\ldots}$ $\mathtt{1cb\ldots}$	$\emptyset$	$\emptyset$	
58	$\mathtt{4\underline{e}e\ldots}$ $\mathtt{3cb\ldots}$	$\emptyset$	$\emptyset$	
59	$\mathtt{5\underline{c}a\ldots}$ $\mathtt{5\_cc\ldots}$	a	$\mathtt{4\underline{e}c\ldots}$	$\mathtt{2\_ec\ldots}$
60	$\mathtt{5\underline{c}a\ldots}$ $\mathtt{3cb\ldots}$	$\emptyset$	$\emptyset$	
61	$\mathtt{5\underline{c}a\ldots}$ $\mathtt{3db\ldots}$	$\emptyset$	$\emptyset$	
62	$\mathtt{5\underline{c}e\ldots}$ $\mathtt{3\_bc\ldots}$	a	$\mathtt{4\underline{e}c\ldots}$	$\mathtt{3cb\ldots}$
63	$\mathtt{5\underline{c}e\ldots}$ $\mathtt{3ca\ldots}$	$\emptyset$	$\emptyset$	
64	$\mathtt{5\underline{e}a\ldots}$ $\mathtt{5\_cc\ldots}$	a	$\mathtt{4\underline{e}e\ldots}$	$\mathtt{2\_ec\ldots}$
65	$\mathtt{5\underline{e}a\ldots}$ $\mathtt{3cb\ldots}$	$\emptyset$	$\emptyset$	
66	$\mathtt{5\underline{e}a\ldots}$ $\mathtt{3db\ldots}$	$\emptyset$	$\emptyset$	
67	$\mathtt{5\underline{e}e\ldots}$ $\mathtt{3\_bc\ldots}$	a	$\mathtt{4\underline{e}e\ldots}$	$\mathtt{3cb\ldots}$
68	$\mathtt{5\underline{e}e\ldots}$ $\mathtt{3ca\ldots}$	$\emptyset$	$\emptyset$	

4 Reformulating the generation of new IRR's from existing ones in terms of IRR generating rules

In order to understand how IRR's can be generated from others, the following IRR of length 6 was chosen from the computer program output and represented in Origin $\to$ LHS $\to$ RHS format as follows:

$$\mathtt{3\underline{a}ecccb} \to \mathtt{1cadbd\underline{b}} \to \mathtt{2dbdbdb\_}. \tag{88}$$

The derivation of the first rule of (88) in single TM steps is

$$\begin{array}{l}
\mathtt{3\underline{a}ecccb} \\
\mathtt{4c\underline{e}cccb} \\
\mathtt{5ca\underline{c}ccb} \\
\mathtt{3caa\underline{c}cb} \\
\mathtt{2ca\underline{a}acb} \\
\mathtt{1cac\underline{a}cb} \\
\mathtt{2ca\underline{c}dcb} \\
\mathtt{1cad\underline{d}cb} \\
\mathtt{2cadb\underline{c}b} \\
\mathtt{1cadbd\underline{b}}
\end{array}. \tag{89}$$

Each time the pointer moves to where it has not been before while going backwards from the LHS, it generates IRR's as follows in triplet and the abbreviated notation:

$$\begin{array}{ll}
\mathtt{2\underline{c}b} \to \mathtt{1d\underline{b}} \to \mathtt{5_cd} \in \mathtt{IRR}(2) & \mathtt{2\underline{c}b} \to\to \mathtt{5_cd} \\
\mathtt{1\underline{d}cb} \to \mathtt{1bd\underline{b}} \to \mathtt{3_ecd} \in \mathtt{IRR}(3) & \mathtt{1\underline{d}cb} \to\to \mathtt{3_ecd} \\
\mathtt{2\underline{c}dcb} \to \mathtt{1dbd\underline{b}} \to \mathtt{5_cecd} \in \mathtt{IRR}(4) & \mathtt{2\underline{c}dcb} \to\to \mathtt{5_cecd} \\
\mathtt{4\underline{e}cccb} \to \mathtt{1adbd\underline{b}} \to \mathtt{2_ececd} \in \mathtt{IRR}(5) & \mathtt{4\underline{e}cccb} \to\to \mathtt{2_ececd}
\end{array}. \tag{90}$$

The abbreviated forms can be obtained by applying in order the following first 3 results to the first of these IRR's $\mathtt{2\underline{c}b} \to\!\!\to \mathtt{5\_cd}$. For the first and third of these steps, there are two options, the first one of which needs to be chosen to obtain (88):

$$\begin{array}{ccc}
\mathtt{2\underline{c}}\ldots \to\!\!\to \mathtt{5_c}\ldots & \stackrel{\mathtt{b}}{\Rightarrow} & \left.\begin{array}{c}\mathtt{1\underline{d}c}\ldots\\ \mathtt{1\underline{e}c}\ldots\end{array}\right\} \to\!\!\to \mathtt{3_ec}\ldots \\
\mathtt{1\underline{d}}\ldots \to\!\!\to \mathtt{3_e}\ldots & \stackrel{\mathtt{d}}{\Rightarrow} & \mathtt{2\underline{c}d}\ldots \to\!\!\to \mathtt{5_ce}\ldots \\
\mathtt{2\underline{c}d}\ldots \to\!\!\to \mathtt{5_ce}\ldots & \stackrel{\mathtt{a}}{\Rightarrow} & \left.\begin{array}{c}\mathtt{4\underline{e}cc}\ldots\\ \mathtt{4\underline{e}ec}\ldots\end{array}\right\} \to\!\!\to \mathtt{2_ece}\ldots \\
\mathtt{4\underline{e}}\ldots \to\!\!\to \mathtt{2_e}\ldots & \stackrel{\mathtt{c}}{\Rightarrow} & \mathtt{3\underline{a}e}\ldots \to\!\!\to \mathtt{1d\underline{e}}\ldots
\end{array} \tag{91}$$

Equation (91) contains examples of IGR's (IRR generating rules) that allow one IRR to be derived from another simply by substituting for the $\ldots$ as the example shows, and express the application of the function $\mathtt{F}$ in Theorem 2.2 in a simpler form. When represented like this, the length of an IGR will be defined as the length of the strings on its RHS, which will always be one more than the length of the strings on its LHS, and will be $\geq \mathtt{2}$. The symbols above the implication signs are the symbol added next to the pointer in the origin (α) in the derivation of the IRR's from other ones given in Section 2, and are the first 4 symbols of the LHS of IRR (88) taken in reverse order. The results on the right in (91) are not necessarily all the results that can be derived from their LHS for that value of α so they could be described as IGR elements. A complete IGR is defined to include all the possible results that can be derived for the number of symbols given for any possible value of α, but if there is not likely to be confusion I will refer to such statements as IGR's.

Because the derivation of the (91) does not involve changing any symbol in the $\ldots$ either on the left or the right of the $\to\!\!\to$, clearly the $\ldots$ on the left of both sides of the statement must be the same, likewise for the right hand sides, but there no connection between them.

The IRR outline on the RHS of the last member of (91) is of type RR so does not generate a new IRR directly. Applying the last member of (91) gives the initial result

$$\mathtt{3\underline{a}ecccb} \to \mathtt{1cadbd\underline{b}} \to \mathtt{1d\underline{e}cecd} \tag{92}$$

which is not an IRR. Taking this computation as far a possible is a result of applying $\mathtt{F}$ to last member of (90) so it has to be an IRR (in this case of non-extendable type RR) which for $\alpha = \mathtt{c}$ is

$$\mathtt{3\underline{a}ecccb} \to\!\!\to \mathtt{2dbdbdb\_} \tag{93}$$

in agreement with (88) (it is straightforward to show that the only other result of applying $\mathtt{F}$ to the last member of (90) is $\mathtt{4\underline{b}ecccb} \to\!\!\to \mathtt{4\_cececd}$ with $\alpha = \mathtt{b}$). This illustrates the general procedure for deriving any IRR by repeated applications of $\mathtt{F}$ starting from a member of IRR(2). Furthermore it classifies each application of $\mathtt{F}$ (an IGR) by an integer $\mathtt{k} \geq 2$ known as its length. It also suggests that there is a procedure for obtaining the IGR's of length $\mathtt{k}$ (members of IGR($\mathtt{k}$)) that when combined could generate some IRR($\mathtt{n}$) for $\mathtt{n} >> \mathtt{k}$.

Consider again in detail the general form of the derivation of an IRR from an existing one. Start with the IRR of type RL

$$\mathtt{t_1\underline{y_1}\ldots y_p} \rightarrow\rightarrow \mathtt{t_2\_z_1 \ldots z_p} \tag{94}$$

where the $\mathtt{t}$'s are machine states and $\mathtt{y}$'s and $\mathtt{z}$'s are symbols. Proceed as usual with $\mathtt{F}$ i.e. add α on the left and right giving $\mathtt{t_1}\alpha\mathtt{\underline{y_1}\ldots y_p} \rightarrow\rightarrow \mathtt{t_2}\underline{\alpha}\mathtt{z_1\ldots z_p}$ and starting from $\mathtt{t_1}\alpha\mathtt{\underline{y_1}\ldots y_p}$ find the new endpoints of the backward search that have the pointer at either end of the extended string that could be none, one or many from the branches of the backward search, and do this for each value of α. For each value of α having at least one endpoint on the left (a new origin), the new RHS is computed. The backward searches can be classified according to the rightmost position (α has position 1) $\mathtt{r_1}$ of the pointer satisfying $\mathtt{2} \leq \mathtt{r_1} \leq \mathtt{p}+\mathtt{1}$. If the endpoint is at the left where the symbol α was, the backward search terminates because the reachability of the LHS has been established (see Lemma 2.1). If the the pointer goes to the right end in the backward search at CS A say then $\mathtt{r_1} = \mathtt{p}+\mathtt{1}$ and the computation can continue backwards from $\mathtt{A}$ because when extra context symbols are added, CS's such as $\mathtt{A}$ can be starting points for further backward searching. These results cannot directly lead to a new IRR and should be included in the set of alternative origins.

If associated with the original IRR (94) there already are some origins of this sort, α is just added on the left to them and no backward searching will give anything new.

If the pointer ends up at the left (a new origin) then clearly we can have $\mathtt{2} \leq \mathtt{r_1} \leq \mathtt{p}$ but we cannot then have $\mathtt{r_1} = \mathtt{p}+\mathtt{1}$. This condition is needed in order to avoid obtaining the IRR outline $\mathtt{Y}$ more than once on the right of the $\Rightarrow$ representing $\mathtt{F}$: in the backward search the pointer cannot reach the opposite end of the string from the α and then return to where the symbol α is. If this did happen, a derivation of the same IRR outline $\mathtt{Y}$ would be obtained from another IRR outline $\mathtt{X_2}$, different from $\mathtt{X_1}$, having as LHS the CS reached when the pointer first reaches the opposite end of the string from the symbol α in the backward search. For example without following this rule it is possible to have two derivations as follows (note the reversal of the orientation here with the α on the right):

$$\left.\begin{array}{l}(\mathtt{X_1})\ \mathtt{4\ldots c\underline{d}} \rightarrow\rightarrow \mathtt{2\ldots db\_} \\ (\mathtt{X_2})\ \mathtt{3\ldots a\underline{c}} \rightarrow\rightarrow \mathtt{2\ldots db\_}\end{array}\right\} \Rightarrow (\mathtt{Y})\ \mathtt{5\ldots cd\underline{d}} \rightarrow\rightarrow \mathtt{1\ldots dbc\_}. \tag{95}$$

This rule would disallow the second derivation. The two IRR outlines on the left have different LHS's which are the middle elements in the triplet of an IRR outline that can usually be not mentioned, and the derivations share much in common as can be seen by single stepping through them.

Analogous to $\mathtt{F}^{-1}$ being unique for IRR's, there is a condition that can be applied to the way $\mathtt{F}$ is obtained to ensure that $\mathtt{F}^{-1}$ is also unique for abbreviated IRR's with the middle element (the LHS) missing from the triplet form (an IRR pair). The above example illustrates this problem.

Theorem 4.1. $\mathtt{F}^{-1}$ *is unique for IRR pairs if in the backward search involved in calculating* $\mathtt{F}$ *for IRR pairs, whenever the pointer reaches the opposite end of the string from the* α *and then returns to where the symbol* α *is, the backward search stops and the resulting CS is ignored.*

Proof. Suppose the following is possible

$$\left.\begin{matrix} \mathtt{A}_1^1 \rightarrow\rightarrow \mathtt{C}_1^0 \\ \mathtt{A}_2^1 \rightarrow\rightarrow \mathtt{C}_2^0 \end{matrix}\right\} \overset{\mathtt{F}}{\Rightarrow} \mathtt{A}_3^1 \rightarrow\rightarrow \mathtt{C}_3^0 \tag{96}$$

where I have used superscripts to represent the position of the pointer in the CS's and the IRR pairs on the LHS have length $\mathtt{n}$. In what follows $\overset{\mathtt{F}}{\Rightarrow}$ will just be written as $\Rightarrow$ as usual. Doing $\mathtt{F}$ starts with adding α on the left which increases all the pointer positions by 1 therefore $\alpha_1\mathtt{A}_1^2 \leftarrow \mathtt{A}_3^1$ and $\alpha_2\mathtt{A}_2^2 \leftarrow \mathtt{A}_3^1$ because $\mathtt{A}_3^1$ is the new origin, and α could be different in the two cases. Therefore doing forward TM computation from $\mathtt{A}_3^1$ both $\alpha_1\mathtt{A}_1^2$ and $\alpha_2\mathtt{A}_2^2$ are reached and we can suppose without loss of generality that $\alpha_1\mathtt{A}_1^2$ is reached first. The pointer can never reach position 1 in these computations because it would imply the pointer reaching position 0 in either of the LHS's of (96) which only happens when one of the CS's $\mathtt{C}$ is reached. Therefore between $\alpha_1\mathtt{A}_1^1$ and $\alpha_2\mathtt{A}_2^1$ the pointer never reaches the α position, so $\alpha_1 = \alpha_2 = \alpha$ say. The first time the pointer gets to position 1 when starting from $\mathtt{A}_3^1$ is when it gets to $\underline{\alpha}\mathtt{C}_1^0$ in the first case and $\underline{\alpha}\mathtt{C}_2^0$ in the second case, and because these are the same computation, $\mathtt{C}_1^0 = \mathtt{C}_2^0$. The computation on the RHS of (96) can now be written as

$$\mathtt{A}_3^1 \rightarrow \alpha\mathtt{A}_1^1 \rightarrow \alpha\mathtt{B}_1^{\mathtt{n}} \rightarrow \alpha\mathtt{A}_2^1 \rightarrow \alpha\mathtt{B}_2^{\mathtt{n}} \rightarrow \underline{\alpha}\mathtt{C}_1^0 \rightarrow \mathtt{C}_3^0 \tag{97}$$

where $\mathtt{B}_1^{\mathtt{n}}$ and $\mathtt{B}_2^{\mathtt{n}}$ are the LHS's in the IRR triplets $\mathtt{Y}_1 = \mathtt{A}_1^1 \rightarrow \mathtt{B}_1^{\mathtt{n}} \rightarrow \mathtt{C}_1^0$ and $\mathtt{Y}_2 = \mathtt{A}_2^1 \rightarrow \mathtt{B}_2^{\mathtt{n}} \rightarrow \mathtt{C}_1^0$ representing the IRR pairs on the LHS of (96). The placement of $\alpha\mathtt{B}_2^{\mathtt{n}}$ is determined by $\mathtt{Y}_2$, and the placement of $\alpha\mathtt{B}_1^{\mathtt{n}}$ cannot be between $\alpha\mathtt{A}_2^1$ and $\underline{\alpha}\mathtt{C}_1^0$ because by lemma 2.1 the origin of $\mathtt{Y}_1$ must be where for the first time the pointer reaches position 1 while going backwards from the LHS which is $\mathtt{B}_1^{\mathtt{n}}$. Then putting α on the left of $\mathtt{Y}_1$ and $\mathtt{Y}_2$ starts the derivation of the RHS of (96) which is (97). This situation resulting from (96) cannot arise if the backward search beyond $\alpha\mathtt{A}_2^1$ reaching $\alpha\mathtt{B}_1^{\mathtt{n}}$ before finally reaching $\mathtt{A}_3^1$ is ignored beyond $\alpha\mathtt{B}_1^{\mathtt{n}}$ i.e. it is as if this CS terminates the search. This is alright because the full form (97) is included in the first part of (96) i.e. $\mathtt{Y}_1 \Rightarrow \mathtt{A}_3^1 \rightarrow\rightarrow \mathtt{C}_3^0$ □

Similarly for the computation of the new RHS, the results can be classified by the rightmost position $\mathtt{r}'_2$ of the pointer. It starts at position 1 and ends at position 0 if it goes left (therefore $\mathtt{1} \le \mathtt{r}'_2 \le \mathtt{p+1}$) and ends at position $\mathtt{p+2}$ if it goes right. The general forms of these results for the new origins are, excluding stationary cycles,

$$\mathtt{t}_1\alpha\underline{\mathtt{y}_1}\dots\mathtt{y}_\mathtt{p} \leftarrow \begin{cases} \mathtt{t}'_1\underline{\alpha}\mathtt{y}'_1\dots\mathtt{y}'_{\mathtt{r}_1-1}\mathtt{y}_{\mathtt{r}_1}\dots\mathtt{y}_\mathtt{p} \text{ where } \mathtt{2} \le \mathtt{r}_1 \le \mathtt{p} \text{ or} \\ \mathtt{t}'_1\alpha\mathtt{y}'_1\dots\mathtt{y}'_{\mathtt{p}-1}\underline{\mathtt{y}_\mathtt{p}} \text{ if } \mathtt{r}_1 = \mathtt{p+1} \end{cases}. \tag{98}$$

Similarly the computation of the new RHS's gives

$$\mathtt{t}_2\underline{\alpha}\mathtt{z}_1\dots\mathtt{z}_\mathtt{p} \rightarrow \begin{cases} \mathtt{t}'_2\_\alpha'\mathtt{z}'_1\dots\mathtt{z}'_{\mathtt{r}'_2-1}\mathtt{z}_{\mathtt{r}'_2}\dots\mathtt{z}_\mathtt{p} \text{ where } \mathtt{1} \le \mathtt{r}'_2 \le \mathtt{p+1} \text{ or} \\ \mathtt{t}'_2\alpha'\mathtt{z}'_1\dots\mathtt{z}'_{\mathtt{p}\_} \text{ if } \mathtt{r}'_2 = \mathtt{p+2} \end{cases}. \tag{99}$$

The reason for the notation $\mathtt{r}'_2$ instead of $\mathtt{r}_2$ is that the minimum number of symbols needed for the representation of the result of applying $\mathtt{F}$ to (94) is

$$\mathtt{r} = \max(\mathtt{r}_1, \mathtt{r}_2) \text{ where } \mathtt{r}_2 = \min(\mathtt{r}'_2, \mathtt{p+1}). \tag{100}$$

If a stationary cycle occurred in (98) it would be noted, but it would have no effect on the general form of the possible results because it would result in a closed circuit in the reverse search path from which paths ending in either type of endpoint in (98) could diverge. If this happened in (99) this result would also be noted, and the absence of an RHS as described would lead to the absence of a new IRR where one was expected.

In (98) and (99) there are 4 cases to be considered because the backward search to get the new origin and forward computation to get the new RHS can each take the pointer to the left or the right independently. Of these, two have $\mathtt{r}_1 = \mathtt{p+1}$ and the result is not an IRR because the LHS is not proved to be reachable. The remaining cases are (1) $\mathtt{2} \le \mathtt{r}_1 \le \mathtt{p}$ and $\mathtt{1} \le \mathtt{r}'_2 = \mathtt{r}_2 \le \mathtt{p+1}$ where the derived IRR is of extendable (RL) type, the derived IRR has the form $\mathtt{1} \rightarrow \mathtt{r}_1 \rightarrow \mathtt{2} \rightarrow \mathtt{p+1} \rightarrow \mathtt{1} \rightarrow \mathtt{r}'_2 = \mathtt{r}_2 \rightarrow \mathtt{0}$, and (2) $\mathtt{2} \le \mathtt{r}_1 \le \mathtt{p}$ and $\mathtt{r}'_2 = \mathtt{p+2}$ (and $\mathtt{r}_2 = \mathtt{p+1}$) where the derived IRR has type RR and the form $\mathtt{1} \rightarrow \mathtt{r}_1 \rightarrow \mathtt{2} \rightarrow \mathtt{p+1} \rightarrow \mathtt{1} \rightarrow \mathtt{p+2}$, where the numbers indicate the pointer position at some extreme points in the computation and α is at position 1. In case (1), Equation (94) leads to the following derived IRR $\mathtt{t}'_1\underline{\alpha}\mathtt{y}'_1\dots\mathtt{y}'_{\mathtt{r}_1-1}\mathtt{y}_{\mathtt{r}_1}\dots\mathtt{y}_\mathtt{p} \rightarrow\rightarrow \mathtt{t}'_2\_\alpha'\mathtt{z}'_1\dots\mathtt{z}'_{\mathtt{r}_2-1}\mathtt{z}_{\mathtt{r}_2}\dots\mathtt{z}_\mathtt{p}$, which can be written if $\mathtt{r}_1 \le \mathtt{r}_2$ without redundant symbols as

$$\begin{array}{l} \mathtt{t}_1\underline{\mathtt{y}_1}\dots\mathtt{y}_{\mathtt{r}_2-1}\dots \rightarrow\rightarrow \mathtt{t}_2\_\mathtt{z}_1\dots\mathtt{z}_{\mathtt{r}_2-1}\dots \Rightarrow \\ \mathtt{t}'_1\underline{\alpha}\mathtt{y}'_1\dots\mathtt{y}'_{\mathtt{r}_1-1}\mathtt{y}_{\mathtt{r}_1}\dots\mathtt{y}_{\mathtt{r}_2-1}\dots \rightarrow\rightarrow \mathtt{t}'_2\_\alpha'\mathtt{z}'_1\dots\mathtt{z}'_{\mathtt{r}_2-1}\dots \end{array} \tag{101}$$

while if $\mathtt{r}_2 \le \mathtt{r}_1$, then similarly the following is obtained:

$$\begin{array}{l} \mathtt{t}_1\underline{\mathtt{y}_1}\dots\mathtt{y}_{\mathtt{r}_1-1}\dots \rightarrow\rightarrow \mathtt{t}_2\_\mathtt{z}_1\dots\mathtt{z}_{\mathtt{r}_1-1}\dots \Rightarrow \\ \mathtt{t}'_1\underline{\alpha}\mathtt{y}'_1\dots\mathtt{y}'_{\mathtt{r}_1-1}\dots \rightarrow\rightarrow \mathtt{t}'_2\_\alpha'\mathtt{z}'_1\dots\mathtt{z}'_{\mathtt{r}_2-1}\mathtt{z}_{\mathtt{r}_2}\dots\mathtt{z}_{\mathtt{r}_1-1}\dots \end{array}. \tag{102}$$

In case (2) the following is obtained:

$$\mathtt{t}_1\underline{\mathtt{y}_1}\ldots\mathtt{y}_\mathtt{p}\rightarrow\rightarrow\mathtt{t}_2\_\mathtt{z}_1\ldots\mathtt{z}_\mathtt{p}\Rightarrow\mathtt{t}'_1\underline{\alpha}\mathtt{y}'_1\ldots\mathtt{y}'_{\mathtt{r}_1-1}\mathtt{y}_{\mathtt{r}_1}\ldots\mathtt{y}_\mathtt{p}\rightarrow\rightarrow\mathtt{t}'_2\alpha'\mathtt{z}'_1\ldots\mathtt{z}'_{\mathtt{p}}\_. \tag{103}$$

For case (1), the result (101) having $\mathtt{r}_2$ symbols on the right and one fewer without the α on the left, is defined to have length $\mathtt{r}_2$. Likewise (102) has length $\mathtt{r}_1$.

The derivation above shows that a single application of $\mathtt{F}$ consists of, in case (1), substituting the symbols specified in (101) or (102) such that the $\ldots$ on the right of the strings in the origins each side of the $\Rightarrow$ are the same, and so are the $\ldots$ on the right of the strings in the RHS's each side of the $\Rightarrow$, but there is no connection between them, and in case (2), applying the result (103). The forms (101), (102) and (103) are defined to be (possibly subsets of) IRR generating rules (IGR's), the complete IGR's having the full list of IRR outlines on the RHS of the $\Rightarrow$ corresponding to the single IRR outline on the left. This full list of results for all values of α all have the same value of $\mathtt{r} \leq \mathtt{p}+\mathtt{1}$ which is the definition of the length of the IGR.

The above argument shows, when combined with Theorem 2.2, that
(1) every IRR of length $\mathtt{p}+\mathtt{1}$ of type RL can be derived by $\mathtt{F}$ from another IRR of length $\mathtt{p}$ of type RL by an IGR with parameters $\mathtt{r}_1$ and $\mathtt{r}_2$ as described and
(2) every IRR of length $\mathtt{p}+\mathtt{1}$ of type RR can be derived by $\mathtt{F}$ from another IRR of length $\mathtt{p}$ of type RL by an IGR with parameter $\mathtt{r}=\mathtt{r}_2=\mathtt{p}+\mathtt{1}$.

As usual, of course the mirror image results of everything in this section hold with the α and $\ldots$ on opposite sides and with right and left exchanged etc.

These can be applied recursively to show that

Theorem 4.2. *any extendable IRR (type RL or LR) of length* $\geq \mathtt{3}$ *can be obtained from a member of IRR(2) by a sequence of substitutions of IGR's as described here under case (1). Any non-extendable IRR (type LL or RR) can be obtained from a member of IRR(2) by the above substitutions (0 or more) followed by a single substitution step under case (2).*

This theorem is illustrated by the example at the beginning of this section.

5 Another look at IGR's and their derivations from others

There were many questions involving the exact definition of IGR's which had to be resolved which has caused some difficulties writing this paper. This was because my ideas were changing during working out many examples. These mainly involved how to handle RHS's with different lengths, excluding redundant symbols, alternative origins, labelling the RHS's with appropriate value of α, multiple origins leading to multiple RHS's (i.e. are they all one IGR or many).

In all that follows, there are right-left reversed forms of IRR outlines and IGR's with α and $\ldots$ on opposite sides of the strings of symbols.

In Section 4 a general description was given of how to find the IGR of minimal length to extend a given extendable IRR of length $\mathtt{p}$ to one of length $\mathtt{p+1}$. This was a reformulation of the process described in Theorem 2.2. Here the main interest is in obtaining IGR's from other ones so as to generate, if possible, the complete set of IGR's for the TM.

Precise definitions of IRR outlines and IGR's are needed to take the next steps. An IRR outline is the set of IRR's with fixed values of the given symbols i.e. $\ldots$ means a string of symbols of arbitrary length including zero. The two occurrences of $\ldots$ can be different but they are of the same length. An IGR is a set of logical implications starting from a fixed IRR outline on the left where in each member a single application of the process in (82) is applied. IGR's from now on will be expressed in terms of IRR outlines. Thus an IGR member is of the form

$$\underline{\mathtt{A}}\ldots \rightarrow\rightarrow \_\mathtt{B}\ldots \Rightarrow \underline{\mathtt{C}}\ldots \rightarrow\rightarrow \mathtt{D}\ldots, \mathtt{S} \tag{104}$$

where each of $\mathtt{A}, \mathtt{B}, \mathtt{C}$ and $\mathtt{D}$ are combinations of machine state, string of symbols, and the pointer position. The pointer positions in $\mathtt{A}$ and $\mathtt{C}$ are at position 1 i.e. the left hand end of the string, and in $\mathtt{B}$ it is to its immediate left as shown, and the pointer position in $\mathtt{D}$ can be immediately to the left or right of the string. If it is to the right the pointer position will not be shown and replaced by $\ldots$. The string components of $\mathtt{A}, \mathtt{B}$ have the same length $\mathtt{p}$, and those of $\mathtt{C}$ and $\mathtt{D}$ have length $\mathtt{p+1}$.

An IRR outline is in general associated with a set $\mathtt{S}$ of alternative origins (with the pointer reaching the opposite end of the string from α) that do not establish the reachability of the LHS in an IRR and $\mathtt{S}$ is only there so that when an extra context symbol is added there is a possibility that an new origin is found that does generate a new IRR which would otherwise have been missed. $\mathtt{S}$ arises only as a result of the abbreviation of IRR's to IRR outlines in the method of applying $\mathtt{F}$ to an IRR outline as described in Table 3 below. It is very useful in the short-cut method of carrying out $\mathtt{F}$ which is a time saving device avoiding duplication of calculations because members of $\mathtt{S}$ are (with an added symbol) possible intermediates in the calculation of new origins from the original ones in the calculation of $\mathtt{F}$, see example 14 in Section 7. If the $\ldots$ to the right of $\Rightarrow$ in an IGR outline are the empty string then any alternative origins that would be in $\mathtt{S}$ are of no further use and could be ignored. If $\mathtt{S}$ is not mentioned it will be assumed that $\mathtt{S} = \emptyset$. In the case that $\mathtt{S} \neq \emptyset$ the values of α must be given alongside the RHS's to which they refer because α is present in each member of $\mathtt{S}$ and can affect the computation of the new origin, and it also determines which RHS is produced.

An IGR will have no redundant symbols where the pointer does not reach during its derivation. This is analogous to IRR's being irreducible. Thus in the derivation of the IGR, the backward search to obtain $\mathtt{C}$ and in the forward computation to obtain $\mathtt{D}$, the pointer never moves outside where strings $\mathtt{C}$ and $\mathtt{D}$ are, except for the last machine step obtaining $\mathtt{D}$ that takes the pointer just outside $\mathtt{D}$. This is in contrast with the derivations of the separate LHS and RHS IRR outlines of the IGR, where the pointer could move further right than this. Therefore the number of symbols retained for each new origin found and RHS must be $\mathtt{r}$ as defined in Equation (100), and any extra symbols used in specific cases will be listed as "context pairs" that can be used to extend the left and right symbol strings respectively on both sides of the logical implication statement as described later (this is not now an IGR). Thus it would be possible for different RHS's of the IGR to have different values of $\mathtt{r}$ corresponding to different values of α, however as will be seen later these will be separated into different IGR's.

Expressing the results of Section 4 in terms of IRR outlines together with contexts to account for the symbols that are not used, gives a more informative description of the algebra than is given on page 23. Specifically, any IRR outline $\mathtt{X}$ of length $\mathtt{p}$ of extendable type can lead under $\mathtt{F}$ to other IRR outlines of length $\mathtt{r}$ of extendable type with parameters $\mathtt{r}_1$ and $\mathtt{r}_2$ as above satisfying $2 \le \mathtt{r}_1 \le \mathtt{p}$ and $1 \le \mathtt{r}_2 \le \mathtt{p}+1$. Thus $2 \le \mathtt{r} \le \mathtt{p}+1$ where $\mathtt{r} = \mathtt{max}(\mathtt{r}_1, \mathtt{r}_2)$ is the length of the derived IRR outline and the IGR. In each of these cases there is a pair of context strings of length $\mathtt{p}+1-\mathtt{r}$ that need to be added to the IRR outline to take the total length of the derived IRR outline to $\mathtt{p}+1$. Alternatively if the derived IRR outline is non-extendable, its length will be $\mathtt{p}+1$.

As a result of applying $\mathtt{F}$ to

$$\mathtt{t}_1\underline{\mathtt{y}_1}\ldots\mathtt{y}_\mathtt{p}\ldots \rightarrow\rightarrow \mathtt{t}_2\_\mathtt{z}_1\ldots\mathtt{z}_\mathtt{p}\ldots,\mathtt{S} \tag{105}$$

first consider the IRR outline. After adding α on the left of the strings in the origin and RHS, there are two possible types of new origin as a result of the backward search from the origin of (105): (1) with the pointer at α and $2 \le \mathtt{r}_1 \le \mathtt{p}$ or (2) with the pointer at $\mathtt{y}_\mathtt{p}$ and $\mathtt{r}_1 = \mathtt{p}+1$. This latter type of result does not allow a proof of the reachability of the LHS so does not lead to an IRR outline (an alternative origin), but it could be the starting point of further computations to get an origin that does so. Let the set of these be $\mathtt{S}^*$ to which $\mathtt{S}$ each member of which has α added on the left i.e. $\alpha\mathtt{S}$ must be added i.e. the new $\mathtt{S}$ is $\mathtt{S}' = \mathtt{S}^* \cup \alpha\mathtt{S}$. This is because in $\alpha\mathtt{S}$ the backward search has no effect because it has already been done to get $\mathtt{S}$. The set $\mathtt{S}$ was given after the IGR in the examples, and is independent of the IRR outline on the right of the IGR and therefore unique for an IGR with multiple RHS's. On the right there are also two types of RHS, (3) with the pointer immediately left of α and $\mathtt{1} \leq \mathtt{r_2} \leq \mathtt{p+1}$ giving an extendable IRR outline, and (4) with the pointer at $\mathtt{z_{p+1}}$ and $\mathtt{r_2} = \mathtt{p+2}$ giving a non-extendable IRR outline. The result of applying $\mathtt{F}$ is then the union over α of all possible pairs of origin under case (1) and RHS, and $\mathtt{S'}$.

One approach that became obvious after studying many examples is to relate the results in Section 4 to the same thing with the context $(\mathtt{y_{p+1}}, \mathtt{z_{p+1}})$ applied to the original IRR i.e. apply $\mathtt{F}$ to

$$\mathtt{t_1\underline{y_1}\ldots y_{p+1}\ldots} \to\!\!\to \mathtt{t_2\_z_1\ldots z_{p+1}\ldots, Sy_{p+1}}. \tag{106}$$

The effect of adding the context on the right of the symbol strings on the origin of (105) is if, $\mathtt{2} \leq \mathtt{r_1} \leq \mathtt{p}$ and the pointer is at α (case 1), just to add $\mathtt{y_{p+1}}$ to the symbol string. If $\mathtt{r_1} = \mathtt{p+1}$ with the pointer at $\mathtt{y_p}$ with $\mathtt{y_{p+1}}$ on its right (case 2), there are now two cases: (5) $\mathtt{y_{p+1}}$ is not reached, $\mathtt{r_1} = \mathtt{p+1}$ and a new origin of IRR's is found, and (6) $\mathtt{y_{p+1}}$ is reached in one step by the pointer and $\mathtt{r_1} = \mathtt{p+2}$ i.e. new alternative origin is found that must be added to the new set $\mathtt{S'}$. Tracing this further back may yield other alternative origins, but by Theorem 4.1 not to a regular origin. Cases 1 and 5 are trivial in that no backward searching is needed. Likewise the symbol $\mathtt{y_{p+1}}$ must also be added to all the old members of $\mathtt{S}$ on the right and the backward search continued to get either case 5 or case 6.

On the right, again there are two cases that are non-trivial (other than when $\mathtt{1} \leq \mathtt{r_2} \leq \mathtt{p+1}$ with the pointer left of α when $\mathtt{z_{p+1}}$ is just added to the string) when pointer reaches $\mathtt{z_{p+1}}$ and $\mathtt{r_2} = \mathtt{p+2}$ (7) the pointer ends up left of α and $\mathtt{r_2} = \mathtt{p+2}$ giving an extendable IRR outline, and (8) the pointer ends up right of $\mathtt{z_{p+1}}$ and $\mathtt{r_2} = \mathtt{p+3}$, giving a non-extendable IRR outline.

Table 3: Generating a new IGR $\mathtt{B}$ from an IGR $\mathtt{A}$ defining the action of $\mathtt{F}$ on IRR $\mathtt{X}$ of type RL and length $\mathtt{p}$ using $\mathtt{X}$ with an extra context pair of symbols.

	Origins			RHS's	
	pointer position	$\mathtt{r_1}$		pointer position	$\mathtt{r_2}$
Case 1	1 i.e. α	$\mathtt{2 \leq r_1 \leq p}$	Case 3	$\mathtt{0}$	$\mathtt{1 \leq r_2 \leq p+1}$
Case 2	$\mathtt{p+1}$ i.e. $\mathtt{y_p}$	$\mathtt{p+1}$	Case 4	$\mathtt{p+2}$	$\mathtt{p+2}$
After adding context $(\mathtt{y_{p+1}}, \mathtt{z_{p+1}})$ to IRR $\mathtt{X}$					
Case 5	$\mathtt{1}$	$\mathtt{p+1}$	Case 7	$\mathtt{0}$	$\mathtt{p+2}$
Case 6	$\mathtt{p+2}$	$\mathtt{p+2}$	Case 8	$\mathtt{p+3}$	$\mathtt{p+3}$

Table 3 summarises the above results, numbering the cases in the same way. The notation α,$\mathtt{r_1}$,$\mathtt{r_2}$,$\mathtt{p}$,$\mathtt{y_p}$,$\mathtt{y_{p+1}}$,$\mathtt{z_{p+1}}$ is as in the development in Section 4. "pointer position" refers to the situation after the application of $\mathtt{F}$, for the origin and RHS, with and without the extra context $(\mathtt{y_{p+1}}, \mathtt{z_{p+1}})$ applied to $\mathtt{X}$ that elongates $\mathtt{X}$ by one symbol in a general way. The cases for the origins and for the RHS's are independent, so you cannot read right across the table and all four combinations 13,14,23,24 of cases are possible after applying $\mathtt{F}$ to the original IRR $\mathtt{X}$. Likewise after adding the context all four combinations

57,58,67,68 are possible, but cases 5 or 6 require first reaching case 2, and cases 7 and 8 require first reaching case 4. In general there may be none or more instances of cases 1,2,5,6 for a fixed `X`, in general dependent on the value of α only for cases 1 and 5. For the RHS's, because forward computation is unique, either case 3 or 4 must occur with only one instance possible in either case for each allowed value of α (i.e. that has an instance of case 1), and likewise for cases 7 and 8 that has an α satisfying case 5.

Because both the forward and backward computations after adding the context can continue, due to the single extra symbol in each case, the results obtained after adding the context depend on the results without the context addition. For the backward search to continue needs case 2, and for the forward computation needs case 4. If neither of these happen, the combination 13, `B` is a trivial extension of `A` by the context and although it is a new extendable IRR it is a trivial extension and will thus not be recorded. The combination 14 gives a non-extendable IRR that is recorded. For combinations 23 and 24 see cases 2,6 below.

Table 3 shows that a consequence of adding the extra context symbol $\mathtt{y_{p+1}}$, in order to get a non-trivial result that needs to be recorded, must be for the pointer to reach this symbol (because the backward search had gone as far as possible without it initially) then go as far as possible in either direction. A similar result holds when doing the forward computations.

Table 4: The meanings of the cases

Cases	meaning
1,5	New origin results demonstrating reachability of the LHS (not shown) which is required for generating a new IRR
2,6	Origins that don't confer reachability to the LHS so don't result in a new IRR. They have α as a parameter. α is not involved in the backward search so the results if any are valid whatever the value of α. Although they do not lead to generation a new IRR, this could happen after application of further contexts. They are listed as the set `S` in the examples as part of the application of `F` for this purpose.
3,7	RHS of a new extendable IRR of type RL.
4,8	RHS of a new non-extendable IRR of type RR.

6 Outline of a possible algorithm for TM analysis

Looking at the many examples of IGR's here including those in the next section suggests the best practical method to analyse the TM is to repeatedly apply `F` to results that generate the IRR(3).

A typical instance involves applying `F` to IRR outline X as follows:

Algorithm 6.1. *[Algorithm for applying* `F` *to an IRR outline* `X`*] Repeat while* `X` *has not already had* `F` *applied to it {Truncate* `X` *by 1 symbol putting the context pair at the end of a sequence or list that starts empty}. Apply* `F` *to* `X` *truncated to length 1 as in the examples above or to the last but one result of this loop by applying the last context pair in the list to each member of* `X` *that is not extendable as its LHS as described in Table 3 and Section 7. Repeat this recursively, unless already done, for any member of* `X` *that is not extendable using the next context pair. Repeat till all the context has been applied. If the result of applying* `F` *is no new results, this terminates this branch of the calculation. The procedure branches whenever there is more than one* `X` *that is of non-extendable type (case 4). The must be a separate pointer to the list of contexts for each branch.*

The intermediate results of this procedure may not be needed in the final analysis. They should be marked as such so that `F` *will not be applied to them at the next stage to generate IGR's for determining the* `IRR(n)`*.*

Theorem 6.2. *Algorithm 6.1 applied to an IRR outline* `X` *generates all the IGRs that define all the results of the application of* `F` *to* `X` *as described in Section 4.*

Proof. (Sketch) This is obvious because truncating `X` truncates both the forward computation and backward search to that length and these calculations must be included in the full calculation for the original `X`. Therefore adding back the effect of the symbols truncated off systematically so that no options are forgotten must give the same result. This holds separately for each individual backward search path and forward computation possible. □

Note that in this algorithm it is the non-extendable IRR outlines on the RHS that can have extra contexts applied to generate new IGR members. If the IRR outlines of the right are extendable they can have `F` applied using a new α to generate new IGR members.

Repeat this starting with each IRR outline generated from the TM until `F` has been applied to every IRR outline generated. All the IGR's so obtained must be recorded including those that lead to no new IRR outlines.

This may terminate because the final result is the IGR used to extend `X` which can often be expected to be much shorter than `X`. See the example in Section 4. I will shortly try to construct a computer program to do this and check against the results obtained so far.

A result of the analysis for a TM will be that each IRR outline has associated with it a set of pairs of context strings that generate the IRR's it corresponds to.

7 A few detailed examples

Example 9 is using up-to-date methods look at this first For example (1) the third member of the RHS of (163) i.e. $\mathtt{2\underline{c}}\beta\ldots \rightarrow\!\!\rightarrow \mathtt{5\_cc}\ldots$ for $\beta \in \{\mathtt{d},\mathtt{e}\}$ is a subset of the (162).4 i.e. $\mathtt{2\underline{c}}\ldots \rightarrow\!\!\rightarrow \mathtt{5\_c}\ldots$ therefore the application of $\mathtt{F}$ to the latter which has already been done is included in the application of $\mathtt{F}$ to the former. To see this note the following, where every step is shown

$$2\alpha\mathtt{\underline{c}d}\ldots \leftarrow \left\{ \begin{array}{l} 1\alpha\mathtt{c\underline{a}}\ldots \leftarrow 2\alpha\mathtt{\underline{a}a}\ldots \left\{\begin{array}{l} \overset{\alpha=\mathtt{b}}{\leftarrow}\left\{\begin{array}{l}\mathtt{1\underline{d}aa}\ldots\\ \mathtt{1\underline{e}aa}\ldots\end{array}\right.\\ \leftarrow 3\alpha\mathtt{a\underline{c}}\ldots\leftarrow \left\{\begin{array}{l} 5\alpha\mathtt{\underline{c}c}\ldots \overset{\alpha=\mathtt{a}}{\leftarrow} \mathtt{4\underline{e}cc}\ldots\\ 5\alpha\mathtt{\underline{e}c}\ldots \overset{\alpha=\mathtt{a}}{\leftarrow}\mathtt{4\underline{e}ec}\ldots\end{array}\right.\end{array}\right.\\ \overset{\alpha=\mathtt{b}}{\leftarrow}\left\{\begin{array}{l}\mathtt{1\underline{d}cd}\ldots\\ \mathtt{1\underline{e}cd}\ldots\end{array}\right.\end{array}\right. \tag{107}$$

and

$$2\alpha\mathtt{\underline{c}e}\ldots \left\{ \begin{array}{l} \leftarrow 5\alpha\mathtt{c\underline{a}}\ldots \leftarrow \emptyset \\ \overset{\alpha=\mathtt{b}}{\leftarrow}\left\{\begin{array}{l}\mathtt{1\underline{d}ce}\ldots\\ \mathtt{1\underline{e}ce}\ldots\end{array}\right. \end{array}\right. . \tag{108}$$

The bottom two branches in (107) and (108) can be deduced from (162).4 so these do not need to be repeated, and the remainder can be summarised as follows

$$2\alpha\mathtt{\underline{c}d}\ldots \left\{ \begin{array}{l} \overset{\alpha=\mathtt{b}}{\leftarrow}\left\{\begin{array}{l}\mathtt{1\underline{d}aa}\ldots\\ \mathtt{1\underline{e}aa}\ldots\end{array}\right.\\ \overset{\alpha=\mathtt{a}}{\leftarrow}\left\{\begin{array}{l}\mathtt{4\underline{e}cc}\ldots\\ \mathtt{4\underline{e}ec}\ldots\end{array}\right. \end{array}\right. \tag{109}$$

because the branch ending in $5\alpha\mathtt{c\underline{a}}\ldots$ represents a case where the backward search ends because $\mathtt{5c\_}$ cannot be arrived at from any TM configuration (see (2)) and the reachability of the LHS is not established so it cannot generate an IRR. After combining (109) with the development of the RHS's this gives the result

$$\mathtt{2\underline{c}}\beta\ldots \rightarrow\!\!\rightarrow \mathtt{5\_cc}\ldots \text{ for } \beta\in\{\mathtt{d},\mathtt{e}\} \Rightarrow \left\{ \begin{array}{l} \left.\begin{array}{l}\mathtt{4\underline{e}cc}\ldots\\ \mathtt{4\underline{e}ec}\ldots\end{array}\right\} \rightarrow\!\!\rightarrow \mathtt{2\_ecc}\ldots \\ \left.\begin{array}{l}\mathtt{1\underline{d}aa}\ldots\\ \mathtt{1\underline{e}aa}\ldots\end{array}\right\} \rightarrow\!\!\rightarrow \mathtt{3\_ecc}\ldots \end{array}\right. \tag{110}$$

additional to (551) with contexts $(\mathtt{d},\mathtt{c})$ and $(\mathtt{e},\mathtt{c})$, which when written out in full are

$$\mathtt{2\underline{c}d}\ldots \rightarrow\!\!\rightarrow \mathtt{5\_cc}\ldots \Rightarrow \left.\begin{array}{l}\mathtt{1\underline{d}cd}\ldots\\ \mathtt{1\underline{e}cd}\ldots\end{array}\right\} \rightarrow\!\!\rightarrow \mathtt{3\_ecc}\ldots \tag{111}$$

and

$$\mathtt{2\underline{c}e}\ldots \rightarrow\!\!\rightarrow \mathtt{5\_cc}\ldots \Rightarrow \left.\begin{array}{l}\mathtt{1\underline{d}ce}\ldots\\ \mathtt{1\underline{e}ce}\ldots\end{array}\right\} \rightarrow\!\!\rightarrow \mathtt{3\_ecc}\ldots . \tag{112}$$

As another example (2) consider the right hand member of (551) which gives rise to the following under $\mathtt{F}$:

$$\mathtt{1}\underline{\gamma}\mathtt{c}\ldots \rightarrow\!\!\rightarrow \mathtt{3\_ec}\ldots \Rightarrow \begin{cases} \mathtt{2}\underline{\mathtt{d}}\gamma\mathtt{c}\ldots \rightarrow\!\!\rightarrow \mathtt{3caa}\ldots \\ \mathtt{2}\underline{\mathtt{a}}\gamma\mathtt{c}\ldots \rightarrow\!\!\rightarrow \mathtt{2\_aec}\ldots \\ \mathtt{2}\underline{\mathtt{c}}\gamma\mathtt{c}\ldots \rightarrow\!\!\rightarrow \mathtt{5\_cec}\ldots \end{cases} \text{for } \gamma \in \{\mathtt{d},\mathtt{e}\} \tag{113}$$

In this case the symbol $\mathtt{c}$ is never reached during the backward search to find new origins and therefore it makes sense to record instead the more general and brief statement of the IGR given that the RHS can be found using just $\mathtt{r} = 2$ symbols, and note the context which is the pair of strings of symbols omitted. The context needs to be specified because it could be involved in the computation of the IGRs giving the IRR's for the next value of $\mathtt{n}$ (5). This gives

$$\mathtt{1}\underline{\gamma}\ldots \rightarrow\!\!\rightarrow \mathtt{3\_e}\ldots \Rightarrow \begin{cases} \mathtt{2}\underline{\mathtt{a}}\gamma\ldots \rightarrow\!\!\rightarrow \mathtt{2\_ae}\ldots \\ \mathtt{2}\underline{\mathtt{c}}\gamma\ldots \rightarrow\!\!\rightarrow \mathtt{5\_ce}\ldots \end{cases} \text{for } \gamma \in \{\mathtt{d},\mathtt{e}\} \text{ context } (\mathtt{c},\mathtt{c}) \tag{114}$$

because it has not already been done above, and the third result derived above must be given in full:

$$\mathtt{1}\underline{\gamma}\mathtt{c}\ldots \rightarrow\!\!\rightarrow \mathtt{3\_ec}\ldots \Rightarrow \mathtt{2}\underline{\mathtt{d}}\gamma\mathtt{c}\ldots \rightarrow\!\!\rightarrow \mathtt{3caa}\ldots. \tag{115}$$

Consider another example (3)

$$\mathtt{3}\ldots\mathtt{b}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{bc\_}. \tag{116}$$

Searching for new origins leads to

$$\mathtt{3}\ldots\mathtt{b}\underline{\mathtt{c}}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} \mathtt{1}\ldots\mathtt{bc}\underline{\mathtt{c}} \\ \overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{4}\ldots\mathtt{bc}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{2}\ldots\mathtt{bc}\underline{\mathtt{e}} \\ \overset{\alpha=\mathtt{e}}{\leftarrow} \mathtt{5}\ldots\mathtt{bc}\underline{\mathtt{b}} \\ \leftarrow \mathtt{3}\ldots\underline{\mathtt{e}}\mathtt{c}\alpha \leftarrow \mathtt{2}\ldots\mathtt{e}\underline{\mathtt{e}}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} \mathtt{3}\ldots\mathtt{ee}\underline{\mathtt{c}} \\ \overset{\alpha=\mathtt{d}}{\leftarrow} \mathtt{1}\ldots\mathtt{ee}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{e}}{\leftarrow} \mathtt{5}\ldots\mathtt{ee}\underline{\mathtt{a}} \end{cases} \end{cases}. \tag{117}$$

Because there is no IGR already with left member $\mathtt{3}\ldots\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{c\_}$, all the results from (117) must be taken into account (otherwise first the 4 branches of the backward search with $\alpha \in \{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{e}\}$ would not have to be done again). Three of these do not involve the symbol $\mathtt{b}$ in the backward search and after completing the RHS's shows that $\mathtt{r} = 2$ so they can be written as

$$\mathtt{3}\ldots\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{c\_} \Rightarrow \begin{cases} \mathtt{1}\ldots\mathtt{c}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{db\_} \\ \mathtt{4}\ldots\mathtt{c}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{db\_} \\ \mathtt{5}\ldots\mathtt{c}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{cb\_} \end{cases} \text{context } (\mathtt{b},\mathtt{b}). \tag{118}$$

The remaining branches give the additional results

$$\mathtt{3}\ldots\mathtt{b\underline{c}} \rightarrow\!\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bc\_} \Rightarrow \begin{cases} \mathtt{2}\ldots\mathtt{bc\underline{e}} \rightarrow\!\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{caa} \\ \mathtt{3}\ldots\mathtt{ee\underline{c}} \rightarrow\!\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{bdb\_} \\ \mathtt{1}\ldots\mathtt{ee\underline{a}} \rightarrow\!\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{bcb\_} \\ \mathtt{5}\ldots\mathtt{ee\underline{a}} \rightarrow\!\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{bcb\_} \end{cases} . \tag{119}$$

Another example (4): start with $\mathtt{5}\ldots\mathtt{d\underline{d}} \rightarrow\!\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bc\_}$. The analysis for $\mathtt{5}\ldots\mathtt{\underline{d}} \rightarrow\!\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{c\_}$ has not been done, so both directions need to be considered initially in the start of the backward search. Only going towards the α gives any results which are new origins $\mathtt{3}\ldots\mathtt{dd\underline{d}}$ and $\mathtt{4}\ldots\mathtt{dd\underline{d}}$ and these only for $\alpha = \mathtt{c}$. The RHS gives $\mathtt{1bc\underline{c}} \rightarrow \mathtt{4\_caa}$ in 3 steps. Computing the RHS used all 3 positions of the pointer , so $\mathtt{r} = \mathtt{3}$ and they will all be displayed in the result which I write as

$$\mathtt{5}\ldots\mathtt{d\underline{d}} \rightarrow\!\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bc\_} \Rightarrow \left.\begin{matrix} \mathtt{3}\ldots\mathtt{dd\underline{d}} \\ \mathtt{4}\ldots\mathtt{dd\underline{d}} \end{matrix}\right\} \rightarrow\!\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{caa}. \tag{120}$$

Another example(5): starting with $\mathtt{3}\ldots\mathtt{d\underline{b}} \rightarrow\!\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bd\_}$, note that the IGR starting with $\mathtt{3}\ldots\mathtt{\underline{b}} \rightarrow\!\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{d\_}$ has not been done. So considering all cases starting the backward search gives

$$\mathtt{3}\ldots\mathtt{d\underline{b}}\alpha \leftarrow \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} \mathtt{1}\ldots\mathtt{db\underline{c}} \\ \overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{4}\ldots\mathtt{db\underline{a}} \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{2}\ldots\mathtt{db\underline{e}} \\ \overset{\alpha=\mathtt{e}}{\leftarrow} \mathtt{5}\ldots\mathtt{db\underline{b}} \end{cases} \tag{121}$$

and the forward computation of the RHS's shows that for $\alpha \in \{\mathtt{b}, \mathtt{c}\}$ all 3 symbol positions on the right are passed by the pointer, and for $\alpha = \mathtt{a}$, 2 symbol positions are used, and for $\alpha = \mathtt{e}$ only 1 position is used. Therefore the 4 results are written with the minimum number of symbols as follows

$$\mathtt{3}\ldots\mathtt{\underline{b}} \rightarrow\!\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{d\_} \Rightarrow \begin{cases} \mathtt{1}\ldots\mathtt{b\underline{c}} \rightarrow\!\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab\_} \\ \mathtt{5}\ldots\mathtt{b\underline{b}} \rightarrow\!\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{db\_} \end{cases} \quad \text{context } (\mathtt{d}, \mathtt{b}) \tag{122}$$

$$\mathtt{3}\ldots\mathtt{d\underline{b}} \rightarrow\!\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bd\_} \Rightarrow \begin{cases} \mathtt{4}\ldots\mathtt{db\underline{a}} \rightarrow\!\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{ecd} \\ \mathtt{2}\ldots\mathtt{db\underline{e}} \rightarrow\!\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{eca} \end{cases} . \tag{123}$$

Example (6): $\mathtt{2\underline{d}d}\ldots \rightarrow\!\!\!\!\rightarrow \mathtt{3\_bc}\ldots$ with context $(\mathtt{e}, \mathtt{c})$. Ignoring the context for now, the backward search gives the following showing every step:

$$2\alpha\mathtt{\underline{d}d}\ldots\left\{\begin{array}{l}\overset{\alpha=\mathtt{b}}{\leftarrow}\left\{\begin{array}{l}\mathtt{1\underline{d}dd}\ldots\\ \mathtt{1\underline{e}dd}\ldots\end{array}\right.\\ \\ \leftarrow 1\alpha\mathtt{d\underline{a}}\ldots\leftarrow 2\alpha\mathtt{\underline{c}a}\ldots\left\{\begin{array}{l}\overset{\alpha=\mathtt{b}}{\leftarrow}\left\{\begin{array}{l}\mathtt{1\underline{d}ca}\ldots\\ \mathtt{1\underline{e}ca}\ldots\end{array}\right.\\ \\ \leftarrow 3\alpha\mathtt{c\underline{c}}\ldots\leftarrow 4\alpha\mathtt{\underline{c}c}\ldots\left\{\begin{array}{l}2\alpha\mathtt{c\underline{b}}\ldots\\ \\ 3\alpha\mathtt{c\underline{b}}\ldots\leftarrow 4\alpha\mathtt{\underline{c}b}\ldots\left\{\begin{array}{l}\overset{\alpha=\mathtt{b}}{\leftarrow}\mathtt{4\underline{b}cb}\ldots\\ \overset{\alpha=\mathtt{c}}{\leftarrow}\mathtt{3\underline{a}cb}\ldots\end{array}\right.\\ \overset{\alpha=\mathtt{b}}{\leftarrow}\mathtt{4\underline{b}cc}\ldots\\ \overset{\alpha=\mathtt{c}}{\leftarrow}\mathtt{3\underline{a}cc}\ldots\end{array}\right.\end{array}\right.\end{array}\right.\tag{124}$$

The effect of adding the context is to allow the possibility of more backward searches from where the pointer reaches the symbol next to .... This gives

$$\begin{array}{l}1\alpha\mathtt{d\underline{a}e}\ldots\leftarrow\emptyset\\ 3\alpha\mathtt{c\underline{c}e}\ldots\leftarrow 5\alpha\mathtt{cc\underline{b}}\ldots\leftarrow\emptyset\\ 2\alpha\mathtt{c\underline{b}e}\ldots\leftarrow\emptyset\\ 3\alpha\mathtt{c\underline{b}e}\ldots\leftarrow 5\alpha\mathtt{cb\underline{b}}\ldots\leftarrow\emptyset\end{array}\tag{125}$$

Because there are no extra origins the result can be summarised as

$$\mathtt{2\underline{d}d}\ldots\rightarrow\rightarrow\mathtt{3\_bc}\ldots\Rightarrow\left\{\begin{array}{l}\left.\begin{array}{l}\mathtt{1\underline{d}dd}\ldots\\ \mathtt{1\underline{e}dd}\ldots\\ \mathtt{1\underline{d}ca}\ldots\\ \mathtt{1\underline{e}ca}\ldots\\ \mathtt{4\underline{b}cc}\ldots\\ \mathtt{4\underline{b}cb}\ldots\end{array}\right\}\rightarrow\rightarrow\mathtt{4\_cbc}\ldots\\ \left.\begin{array}{l}\mathtt{3\underline{a}cc}\ldots\\ \mathtt{3\underline{a}cb}\ldots\end{array}\right\}\rightarrow\rightarrow\mathtt{2\_abc}\ldots\end{array}\right.\qquad\text{context }(\mathtt{e},\mathtt{c})\tag{126}$$

and further written in terms of earlier results as

$$(243)\text{ context }(\mathtt{e},\mathtt{c})\tag{127}$$

and

$$(173)\text{ context }(\mathtt{d},\mathtt{c})\text{ context }(\mathtt{e},\mathtt{c}).\tag{128}$$

Because the contexts are written on the right (where the ... are) in (128) the contexts can be combined to give

$$(173)\text{ context }(\mathtt{de},\mathtt{cc}).\tag{129}$$

otherwise they would have been combined in reverse order.

Example (7): consider $\mathtt{X} = \mathtt{2\underline{c}e}\ldots \rightarrow\!\!\rightarrow \mathtt{5\_cc}\ldots$ with context $(\mathtt{e},\mathtt{c})$. Remove the rightmost symbols to get $\mathtt{Y} = \mathtt{2\underline{c}}\ldots \rightarrow\!\!\rightarrow \mathtt{5\_c}\ldots$. $\mathtt{F}$ has been applied to $\mathtt{Y}$ giving results (168) that generate only extendable IRR's therefore contextual symbols can be added generating results of the same type in this case from $\mathtt{X}$, $\mathtt{F}$ gives (168) with context $(\mathtt{e},\mathtt{c})$, to which an extra context $(\mathtt{e},\mathtt{c})$ is needed. In addition, results can possibly be found starting the backward search from $\mathtt{2}\alpha\mathtt{\underline{c}e}\ldots$ where the first backward step is to the right giving $\mathtt{2}\alpha\mathtt{\underline{c}e}\ldots \leftarrow \mathtt{5}\alpha\mathtt{c\underline{a}}\ldots \leftarrow \emptyset$ giving no new origins. Adding the final context symbol and continuing the backward search gives $\mathtt{5}\alpha\mathtt{c\underline{a}e}\ldots \leftarrow \emptyset$ so $\mathtt{F}$ applied to $\mathtt{X}$ context $(\mathtt{e},\mathtt{c})$ gives only (168) with context $(\mathtt{ee},\mathtt{cc})$.

Example (8): consider $\mathtt{X} = \mathtt{1\underline{d}a}\ldots \rightarrow\!\!\rightarrow \mathtt{4\_ca}\ldots$ with context $(\mathtt{c},\mathtt{a})$. Removing the rightmost symbols gives $\mathtt{Y} = \mathtt{1\underline{d}}\ldots \rightarrow\!\!\rightarrow \mathtt{4\_c}\ldots$ which has been done in (163) giving 3 results, one generating non-extendable IRR's, but $\mathtt{X}$ has not been done. Doing it now (applying $\mathtt{F}$ to $\mathtt{X}$) starts with the backward search $\mathtt{1}\alpha\mathtt{\underline{d}a}\ldots \leftarrow \emptyset$, but in addition there is the result of adding the context symbols $(\mathtt{a},\mathtt{a})$ back to the result of applying $\mathtt{F}$ to $\mathtt{Y}$ giving (using $\mathtt{2ab\underline{a}} \rightarrow \mathtt{1abc\_}$)

$$\mathtt{1\underline{d}a}\ldots \rightarrow\!\!\rightarrow \mathtt{4\_ca}\ldots \Rightarrow \begin{cases} \mathtt{2\underline{d}da}\ldots \rightarrow\!\!\rightarrow \mathtt{3\_bca}\ldots \\ \mathtt{2\underline{a}da}\ldots \rightarrow\!\!\rightarrow \mathtt{1abc}\ldots \\ \mathtt{2\underline{c}da}\ldots \rightarrow\!\!\rightarrow \mathtt{5\_cca}\ldots \end{cases} \tag{130}$$

which is probably better expressed by the two equations following:

$$\mathtt{1\underline{d}}\ldots \rightarrow\!\!\rightarrow \mathtt{4\_c}\ldots \Rightarrow \begin{cases} \mathtt{2\underline{d}d}\ldots \rightarrow\!\!\rightarrow \mathtt{3\_bc}\ldots \\ \mathtt{2\underline{c}d}\ldots \rightarrow\!\!\rightarrow \mathtt{5\_cc}\ldots \end{cases} \tag{131}$$

with context $(\mathtt{a},\mathtt{a})$ and

$$\mathtt{1\underline{d}a}\ldots \rightarrow\!\!\rightarrow \mathtt{4\_ca}\ldots \Rightarrow \mathtt{2\underline{a}da}\ldots \rightarrow\!\!\rightarrow \mathtt{1abc}\ldots \tag{132}$$

Now including the additional context $(\mathtt{c},\mathtt{a})$ under $\mathtt{F}$ gives

$$(131)\ \text{context}\ (\mathtt{ac},\mathtt{aa}) \tag{133}$$

and (using $\mathtt{1abc\underline{a}} \rightarrow \mathtt{2abdb\_}$)

$$\mathtt{1\underline{d}ac}\ldots \rightarrow\!\!\rightarrow \mathtt{4\_caa}\ldots \Rightarrow \mathtt{2\underline{a}dac}\ldots \rightarrow\!\!\rightarrow \mathtt{2abdb}\ldots \tag{134}$$

Example (9):

$\mathtt{EX}()$ is the function that picks out the extendable subset of an IGR. Apply $\mathtt{F}$ to the RHS's of (177) with context $(\mathtt{d},\mathtt{c})$. Consider first

$$\mathtt{F}\left(\left\{ \begin{array}{l} \mathtt{5\underline{c}e}\ldots \\ \mathtt{5\underline{e}e}\ldots \end{array} \right\} \rightarrow\!\!\rightarrow \mathtt{5ca}\ldots + (\mathtt{d},\mathtt{c})\right) = \emptyset \tag{135}$$

using Theorem **??** because

$$\mathtt{F}\left(\left\{ \begin{array}{l} \mathtt{5\underline{c}e}\ldots \\ \mathtt{5\underline{e}e}\ldots \end{array} \right\} \rightarrow\!\!\rightarrow \mathtt{5ca}\ldots\right) = \emptyset \tag{136}$$

and $5\alpha\underline{\mathtt{c}}\mathtt{e}\ldots \leftarrow \emptyset$ and $5\alpha\underline{\mathtt{e}}\mathtt{e}\ldots \leftarrow \emptyset$, where the first backward step is restricted to being to the right in each case.

Next consider $\mathtt{F}(3\underline{\mathtt{e}}\mathtt{e}\ldots \twoheadrightarrow 4\_\mathtt{ce}\ldots + (\mathtt{d},\mathtt{c})) = \mathtt{F}(\mathtt{LHS}(179) + (\mathtt{ed},\mathtt{ec})) = \mathtt{RHS}(179) + (\mathtt{ed},\mathtt{ec})$ using Theorem **??** and the fact that $3\alpha\underline{\mathtt{e}}\mathtt{ed}\ldots \leftarrow 5\alpha\mathtt{e}\underline{\mathtt{b}}\mathtt{d}\ldots \leftarrow \emptyset$ where the backward search path with the pointer moving to the left at the first reverse step was omitted because the pointer there does not reach the string $\mathtt{p}$ which is $\mathtt{ed}$. This is the same as

$$\text{RHS's of (179) context } (\mathtt{e},\mathtt{e}) \text{ context } (\mathtt{d},\mathtt{c}). \tag{137}$$

The cases of this with the pointer moving unexpectedly (the others are trivial to write down) are $3\underline{\mathtt{e}}\mathtt{e}\ldots \twoheadrightarrow 3\mathtt{bc}\ldots$ and $4\underline{\mathtt{c}}\mathtt{e}\ldots \twoheadrightarrow 2\mathtt{ab}\ldots$ to which first context $(\mathtt{e},\mathtt{e})$ is applied, then $(\mathtt{d},\mathtt{c})$. The first stage gives

$$3\underline{\mathtt{e}}\mathtt{ee}\ldots \twoheadrightarrow 3\mathtt{bcb}\ldots \tag{138}$$

using $3\mathtt{bc}\underline{\mathtt{e}}\ldots \to 3\mathtt{bcb}\ldots$ and

$$4\underline{\mathtt{c}}\mathtt{ee}\ldots \twoheadrightarrow 3\_\mathtt{bcc}\ldots \tag{139}$$

respectively (using $2\mathtt{ab}\underline{\mathtt{e}}\ldots \to 3\_\mathtt{bcc}\ldots$ from (46)) , and applying $(\mathtt{d},\mathtt{c})$ to the first of these gives

$$3\underline{\mathtt{e}}\mathtt{eed}\ldots \twoheadrightarrow 1\mathtt{babc}\ldots \tag{140}$$

(using $3\mathtt{bcb}\underline{\mathtt{c}}\ldots \to 1\mathtt{babc}\ldots$ from (60)). Therefore the IGR's giving these results in general form expressed with the minimum number of symbols are $\mathtt{EX}(179)$ context $(\mathtt{ed},\mathtt{ec})$, $\mathtt{EX}(389)$ context $(\mathtt{d},\mathtt{c})$, and $3\underline{\mathtt{e}}\mathtt{ed}\ldots \twoheadrightarrow 4\_\mathtt{cec}\ldots \Rightarrow 3\underline{\mathtt{e}}\mathtt{eed}\ldots \twoheadrightarrow 1\mathtt{babc}\ldots$, which can be written more explicitly as

$$\left\{\begin{array}{l} 3\underline{\mathtt{e}}\ldots \twoheadrightarrow 4\_\mathtt{c}\ldots \Rightarrow \left.\begin{array}{c} 5\underline{\mathtt{c}}\mathtt{e}\ldots \\ 5\underline{\mathtt{e}}\mathtt{e}\ldots \end{array}\right\} \twoheadrightarrow 3\_\mathtt{bc}\ldots \text{ context } (\mathtt{ed},\mathtt{ec}) \\ 3\underline{\mathtt{e}}\mathtt{ed}\ldots \twoheadrightarrow 4\_\mathtt{cec}\ldots \Rightarrow 3\underline{\mathtt{e}}\mathtt{eed}\ldots \twoheadrightarrow 1\mathtt{babc}\ldots \\ 3\underline{\mathtt{e}}\mathtt{e}\ldots \twoheadrightarrow 4\_\mathtt{ce}\ldots \Rightarrow 4\underline{\mathtt{c}}\mathtt{ee}\ldots \twoheadrightarrow 3\_\mathtt{bcc}\ldots \text{ context } (\mathtt{d},\mathtt{c}) \end{array}\right. . \tag{141}$$

This form is most useful because for example in the last line the pointer does not reach the rightmost symbol (in the context) in its derivation, which is more informative than just $3\underline{\mathtt{e}}\mathtt{ed}\ldots \twoheadrightarrow 4\_\mathtt{cec}\ldots \Rightarrow 4\underline{\mathtt{c}}\mathtt{eed}\ldots \twoheadrightarrow 3\_\mathtt{bccc}\ldots$ in which the pointer could apparently have reached the last symbol if the derivation was not known.

Next consider $\mathtt{F}(4\underline{\mathtt{c}}\mathtt{e}\ldots \twoheadrightarrow 2\_\mathtt{ae}\ldots + (\mathtt{d},\mathtt{c}))$ gives just

$$(258) \text{ context } (\mathtt{ed},\mathtt{ec}) \tag{142}$$

because $4\alpha\underline{\mathtt{c}}\mathtt{ed}\ldots \leftarrow \emptyset$ under the condition that the first backward TM step in the backward search is to the right.

Example 10.

From the first RHS of (163) i.e. $\mathtt{2\underline{d}d}\ldots \to\to \mathtt{3\_bc}\ldots$ context $(\mathtt{e},\mathtt{c})$, writing this out in full and applying $\mathtt{F}$ leads to a set of results that can be classified by $\mathtt{r_1}$ and $\mathtt{r_2}$. They can then be written without any redundant symbols and the use of contexts as follows:

$$\mathtt{2\underline{d}}\ldots \to\to \mathtt{3\_b}\ldots \Rightarrow \left.\begin{array}{l}\mathtt{1\underline{d}d}\ldots\\ \mathtt{1\underline{e}d}\ldots\end{array}\right\} \to\to \mathtt{4\_cb}\ldots \ \text{context } (\mathtt{de},\mathtt{cc}) \tag{143}$$

$$\mathtt{2\underline{d}d}\ldots \to\to \mathtt{3\_bc}\ldots \Rightarrow \left\{\begin{array}{l} \left.\begin{array}{l}\mathtt{1\underline{d}ca}\ldots\\ \mathtt{1\underline{e}ca}\ldots\\ \mathtt{4\underline{b}ca}\ldots\\ \mathtt{4\underline{b}cb}\ldots\end{array}\right\} \to\to \mathtt{4\_cbc} \\ \left.\begin{array}{l}\mathtt{3\underline{a}cc}\ldots\\ \mathtt{3\underline{a}cb}\ldots\end{array}\right\} \to\to \mathtt{2\_abc}\ldots \end{array}\right. \qquad \text{context } (\mathtt{e},\mathtt{c}). \tag{144}$$

These were obtained from the backward search starting from $\mathtt{2}\alpha\mathtt{\underline{d}de}\ldots$ giving 8 results which had respectively two with $\mathtt{r_1} = \mathtt{2}$, four with $\mathtt{r_1} = \mathtt{3}$ each with $\alpha = \mathtt{b}$, and the last two with $\alpha = \mathtt{c}$ and $\mathtt{r_1} = \mathtt{3}$.

so for example in deriving

$$\mathtt{4\underline{c}e}\ldots \to\to \mathtt{2\_ae}\ldots \Rightarrow \mathtt{3\underline{a}ce}\ldots \to\to \mathtt{3\_bcc}\ldots \tag{145}$$

the first reverse move of the pointer is towards the α indicating that it should be derived from the IGR with the rightmost symbols removed (which involves the backward search starting by moving towards the α)

$$\mathtt{4\underline{c}}\ldots \to\to \mathtt{2\_a}\ldots \Rightarrow \mathtt{3\underline{a}c}\ldots \to\to \mathtt{2ab}\ldots \tag{146}$$

followed by putting the symbol on the right back giving $\mathtt{2ab\underline{e}}\ldots \to \mathtt{3\_bcc}\ldots$. The method in example 10 could be carried out for all the other results, but it would likely be very slow and inefficient. Note that $\mathtt{2\underline{d}}\ldots \to\to \mathtt{3\_b}\ldots$ has already had $\mathtt{F}$ done to it.

$$\mathtt{5}\ldots\mathtt{dc\underline{b}} \to\to \mathtt{1}\ldots\mathtt{abd\_} \tag{147}$$

needs to have $\mathtt{F}$ applied to it. In this case not even $\mathtt{5}\ldots\mathtt{\underline{b}} \to\to \mathtt{1}\ldots\mathtt{d\_}$ has yet had $\mathtt{F}$ applied to it, so this is done next giving

$$\mathtt{5}\ldots\mathtt{\underline{b}} \to\to \mathtt{1}\ldots\mathtt{d\_} \Rightarrow \left.\begin{array}{l}\mathtt{3}\ldots\mathtt{b\underline{d}}\\ \mathtt{4}\ldots\mathtt{b\underline{d}}\end{array}\right\} \to\to \mathtt{5}\ldots\mathtt{ca}. \tag{148}$$

Applying the context $(\mathtt{c},\mathtt{b})$ gives no backward search paths that take the pointer to the $\mathtt{c}$ so we just get

$$\mathtt{5}\ldots\mathtt{c\underline{b}} \to\to \mathtt{1}\ldots\mathtt{bd\_} \Rightarrow \left.\begin{array}{l}\mathtt{3}\ldots\mathtt{cb\underline{d}}\\ \mathtt{4}\ldots\mathtt{cb\underline{d}}\end{array}\right\} \to\to \mathtt{3}\ldots\mathtt{eca}. \tag{149}$$

This would not have to be written down ($\mathtt{r_1} = \mathtt{2}$) if it wasn't for the fact that on the right the pointer again ends up at the left (within the $\ldots$) and in fact $\mathtt{r_2} = \mathtt{3}$ giving $\mathtt{r} = \mathtt{3}$. This illustrates the fact that the result needs to be written if and only if $\mathtt{r} > \mathtt{p}$ (otherwise it is just a trivial extension of a previous IRR outline). Now applying the final context $(\mathtt{d}, \mathtt{a})$ gives

$$\mathtt{5}\ldots\mathtt{dc\underline{b}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{abd\_} \Rightarrow \left.\begin{array}{l}\mathtt{3}\ldots\mathtt{dcb\underline{d}}\\ \mathtt{4}\ldots\mathtt{dcb\underline{d}}\end{array}\right\} \rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{caac\_}. \tag{150}$$

Here $\mathtt{r_1} = \mathtt{2}$ and $\mathtt{r_2} = \mathtt{4}$.

Example 12

$\mathtt{F}$ needs to be applied to $\mathtt{5}\ldots\mathtt{aa\underline{d}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{abc\_}$, and it has not already been applied to $\mathtt{5}\ldots\mathtt{a\underline{d}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{bc\_}$ nor to $\mathtt{5}\ldots\mathtt{\underline{d}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{c\_}$, so the context pairs $(\mathtt{a}, \mathtt{a})$ and $(\mathtt{a}, \mathtt{b})$ removed in these steps will be kept for later. Now $\mathtt{5}\ldots\mathtt{\underline{d}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{c\_}$ needs to have $\mathtt{F}$ applied to it. This gives

$$\mathtt{5}\ldots\mathtt{\underline{d}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{c\_} \Rightarrow \left.\begin{array}{l}\mathtt{3}\ldots\mathtt{d\underline{d}}\\ \mathtt{4}\ldots\mathtt{d\underline{d}}\end{array}\right\} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab\_}. \tag{151}$$

Applying the context $(\mathtt{a}, \mathtt{b})$ gives, apart from the trivial result which leads to the above with context $(\mathtt{a}, \mathtt{b})$ applied, the alternative origin $\mathtt{4}\ldots\mathtt{\underline{e}d}\alpha$ because on the left the backward search starting from $\mathtt{5}\ldots\mathtt{a\underline{d}}\alpha \leftarrow \mathtt{4}\ldots\mathtt{\underline{e}d}\alpha \leftarrow \mathtt{1}\ldots\mathtt{e\underline{b}}\alpha \leftarrow \emptyset$. Next applying the remaining context $(\mathtt{a}, \mathtt{a})$ gives no non-trivial results because $\mathtt{4}\ldots\mathtt{a\underline{e}d}\alpha \leftarrow \mathtt{1}\ldots\mathtt{ae\underline{b}}\alpha \leftarrow \emptyset$. and it is unnecessary to compute the RHS.

Example 13

Again for applying $\mathtt{F}$ to $\mathtt{3}\ldots\mathtt{eb\underline{d}} \rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{bcc\_}$, $\mathtt{F}$ has not already been applied to it or either of its shortened forms by deleting symbols from the left. Starting from

$$\mathtt{3}\ldots\mathtt{\underline{d}} \rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{c\_} \Rightarrow \left\{\begin{array}{l}\mathtt{a}: \mathtt{1}\ldots\mathtt{d\underline{c}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab}\\ \mathtt{b}: \mathtt{4}\ldots\mathtt{d\underline{a}} \rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{cb\_}\\ \mathtt{c}: \mathtt{2}\ldots\mathtt{d\underline{e}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{cc\_}\\ \mathtt{e}: \mathtt{5}\ldots\mathtt{d\underline{b}} \rightarrow\!\!\rightarrow \mathtt{5}\ldots\mathtt{ca\_}\end{array}\right. \tag{152}$$

only the first of these results is non-extendable by $\mathtt{F}$ and leading to possibly non-trivial results of putting back 1 pair of context symbols. The next paragraph shows why the α values are in general needed.

Adding back the context $(\mathtt{b}, \mathtt{c})$ gives $\mathtt{3}\ldots\mathtt{b\underline{d}} \rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{cc\_}$ on the LHS, and the backward search for each value of α i.e. $\alpha \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{e}\}$ starts from $\mathtt{3}\ldots\mathtt{b\underline{d}}\alpha$ giving one branch as

$$\mathtt{3}\ldots\mathtt{b\underline{d}}\alpha \leftarrow \mathtt{3}\ldots\mathtt{\underline{e}d}\alpha \leftarrow \emptyset \tag{153}$$

therefore $\mathtt{3}\ldots\mathtt{\underline{e}d}\alpha$ is a new alternative origin that needs to be recorded because if another symbol is added by the $\ldots$, the backward search could possibly continue. The only other branches of the backward search are single steps to the right analogous to what was done to get (321) and generate results with $\mathtt{r}_1 = 2$. On the right only for $\alpha = \mathtt{a}$ is the new RHS non-trivial because $2\ldots\underline{\mathtt{c}}\mathtt{ab} \to 3\ldots\mathtt{abc\_}$ giving $\mathtt{r}_2 = 3$, with the other cases having $\mathtt{r}_2 = 2$. For each value of α there is thus a set of origins possible and an RHS, determining $\mathtt{r}_1, \mathtt{r}_2$ and $\mathtt{r} = \max(\mathtt{r}_1, \mathtt{r}_2)$ in each case. If $\mathtt{r} = 2$ the effect of the addition of the context is just to add the context symbols on the LHS and and RHS of the $\to\to$ respectively, on both sides of the $\Rightarrow$ and will not be recorded, therefore the only results recorded are either from the alternative origin (that applies to all values of α), or from the case $\alpha = \mathtt{a}$ which is non-extendable by $\mathtt{F}$ and has the pointer position not shown in the RHS. The α value could be needed in another extension by a context because that value will have to be added on the right to get a new RHS. Therefore I write the result as

$$3\ldots\mathtt{b}\underline{\mathtt{d}} \to\to 4\ldots\mathtt{cc\_} \Rightarrow \mathtt{a} : 1\ldots\mathtt{bd}\underline{\mathtt{c}} \to\to 3\ldots\mathtt{abc\_}, \{3\ldots\underline{\mathtt{e}}\mathtt{da}\} \tag{154}$$

The last context to be applied is $(\mathtt{e}, \mathtt{b})$. Adding the context symbol to the alternative origin and start the backward search from there gives $3\ldots\mathtt{e}\underline{\mathtt{e}}\mathtt{da} \leftarrow \emptyset$. Therefore the final result is

$$3\ldots\mathtt{eb}\underline{\mathtt{d}} \to\to 4\ldots\mathtt{bcc\_} \Rightarrow \emptyset, \emptyset \tag{155}$$

other than the trivial extension of (322) by the context which adds nothing new.

Example 14

Consider applying the context $(\mathtt{e}, \mathtt{c})$ to (240) to generate the RHS of $2\underline{\mathtt{c}}\mathtt{de}\ldots \to\to 5\mathtt{\_ccc}\ldots \Rightarrow$. I chose this example because this at first sight seems very complex and it illustrates many concepts.

Firstly it is of course possible to carry out this calculation according to the method described in Section 4. But doing so will inevitably repeat much of what was done in the derivation of (240) with an extra symbol added. Hence the motivation for the short cut method described in Section 5.

The first step is to add in the context symbol $\mathtt{e}$ to the alternative origins and start the backward search from each case. The initial backward step must be restricted(R) to going to the added symbol which happens to be on the right here but could be to the left in other examples. This is indicated by $\overset{\mathtt{R}}{\leftarrow}$ and avoids repetition from earlier calculations. These give the extra results from adding the extra symbol $\mathtt{e}$ thus

$$\begin{aligned}
&\mathtt{1}\alpha\mathtt{c\underline{a}e}\ldots \overset{\mathtt{R}}{\leftarrow} \emptyset\\
&\mathtt{1}\alpha\mathtt{c\underline{b}e}\ldots \overset{\mathtt{R}}{\leftarrow} \emptyset\\
&\mathtt{3}\alpha\mathtt{a\underline{c}e}\ldots \overset{\mathtt{R}}{\leftarrow} \mathtt{5}\alpha\mathtt{ac\underline{b}}\ldots \leftarrow \emptyset\\
&\mathtt{3}\alpha\mathtt{c\underline{d}e}\ldots \overset{\mathtt{R}}{\leftarrow} \mathtt{5}\alpha\mathtt{cd\underline{b}}\ldots \leftarrow \emptyset\\
&\mathtt{4}\alpha\mathtt{c\underline{d}e}\ldots \overset{\mathtt{R}}{\leftarrow} \emptyset\\
&\mathtt{3}\alpha\mathtt{e\underline{d}e}\ldots \overset{\mathtt{R}}{\leftarrow} \mathtt{5}\alpha\mathtt{ed\underline{b}}\ldots \leftarrow \emptyset\\
&\mathtt{4}\alpha\mathtt{e\underline{d}e}\ldots \overset{\mathtt{R}}{\leftarrow} \emptyset
\end{aligned} \tag{156}$$

This establishes the new alternative origins as $\{\mathtt{5}\alpha\mathtt{ac\underline{b}}\ldots, \mathtt{5}\alpha\mathtt{cd\underline{b}}\ldots, \mathtt{5}\alpha\mathtt{ed\underline{b}}\ldots\}$ and shows that there are no new regular origins.

The results must be obtained separately for each value of $\alpha \in \{\mathtt{a},\mathtt{b},\mathtt{c}\}$ involved in (240), and recall that $\mathtt{r} = \max(\mathtt{r}_1, \mathtt{r}_2) = \mathtt{4}$ is a necessary condition for a result to be recorded, otherwise it is a trivial extension of an IGR for $\mathtt{r} \leq \mathtt{3}$ obtained by adding the context $(\mathtt{e},\mathtt{c})$ to (240). For $\alpha \in \{\mathtt{a},\mathtt{b}\}$, the RHS's with the symbol $\mathtt{c}$ added are $\mathtt{2\_eccc}\ldots$ and $\mathtt{3\_eccc}\ldots$ for $\alpha = \mathtt{a},\mathtt{b}$ respectively. These RHS's are trivially obtained and have no TM steps $\mathtt{r}_2$ should be defined as 0. Also there are no new regular origins above that would have had $\mathtt{r}_1 = \mathtt{4}$ because the pointer would have passed all four places in the string to end up at position 0, therefore in these cases $\mathtt{r} < \mathtt{4}$ giving nothing non-trivial. For $\alpha = \mathtt{c}$ the RHS is obtained from $\mathtt{1dbd\underline{c}}\ldots \to \mathtt{5\_ceca}\ldots$ giving $\mathtt{r}_2 = \mathtt{r} = \mathtt{4}$ therefore all the origins for this case must be worked out which are trivially $\mathtt{3\underline{a}cde}\ldots$ and $\mathtt{4\underline{c}ade}\ldots$ by just adding in the symbol $\mathtt{e}$ on the right. The alternative origins must also be included giving finally

$$\mathtt{2\underline{c}de}\ldots \to\!\!\to \mathtt{5\_ccc}\ldots \Rightarrow \mathtt{c}: \left.\begin{array}{l}\mathtt{3\underline{a}cde}\ldots\\ \mathtt{4\underline{c}ade}\ldots\end{array}\right\} \to\!\!\to \mathtt{5\_ceca}\ldots, \left\{\begin{array}{l}\mathtt{5cac\underline{b}}\ldots\\ \mathtt{5ccd\underline{b}}\ldots\\ \mathtt{5ced\underline{b}}\ldots\end{array}\right\} \tag{157}$$

Example 15

Consider the case (246). Adding the extra symbol to the alternative origin and tracing back gives

$$\mathtt{4}\alpha\mathtt{a\underline{a}c}\ldots \overset{\mathtt{R}}{\leftarrow} \left\{\begin{array}{l}
\mathtt{2}\alpha\mathtt{aa\underline{b}}\ldots \leftarrow \emptyset\\[2ex]
\mathtt{3}\alpha\mathtt{aa\underline{b}}\ldots \leftarrow \left\{\begin{array}{l}
\mathtt{5}\alpha\mathtt{a\underline{c}b}\ldots \leftarrow \mathtt{4}\alpha\mathtt{\underline{e}cb}\ldots \left\{\begin{array}{l}
\leftarrow \mathtt{2}\alpha\mathtt{e\underline{b}b}\ldots \leftarrow \emptyset\\
\leftarrow \mathtt{3}\alpha\mathtt{e\underline{b}b}\ldots \leftarrow \emptyset\\
\overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{4\underline{b}ecb}\ldots\\
\overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{3\underline{a}ecb}\ldots
\end{array}\right.\\[4ex]
\mathtt{5}\alpha\mathtt{a\underline{e}b}\ldots \leftarrow \mathtt{4}\alpha\mathtt{\underline{e}eb}\ldots \left\{\begin{array}{l}
\overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{4\underline{b}eeb}\ldots\\
\overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{3\underline{a}eeb}\ldots
\end{array}\right.
\end{array}\right.
\end{array}\right. \tag{158}$$

This gives two origins for each of $\alpha \in \{\texttt{b}, \texttt{c}\}$ and two new alternative origins. The RHS's for $\alpha \in \{\texttt{a}, \texttt{c}\}$ give $\texttt{r}_2 = \texttt{0}$. Therefore $\texttt{r} = \texttt{4}$ for $\alpha \in \{\texttt{b}, \texttt{c}\}$ only and the results are

$$\texttt{3\underline{a}bd}\ldots \rightarrow\!\!\rightarrow \texttt{2\_aec}\ldots \Rightarrow \left\{ \begin{array}{ll} \texttt{b}: & \left.\begin{array}{l} \texttt{4\underline{b}ecb}\ldots \\ \texttt{4\underline{b}eeb}\ldots \end{array}\right\} \rightarrow\!\!\rightarrow \texttt{4\_caec}\ldots, \left\{\begin{array}{l} \texttt{2baa\underline{b}}\ldots \\ \texttt{3baa\underline{b}}\ldots \end{array}\right\} \\ \texttt{c}: & \left.\begin{array}{l} \texttt{3\underline{a}ecb}\ldots \\ \texttt{3\underline{a}eeb}\ldots \\ \texttt{4\underline{c}abd}\ldots \end{array}\right\} \rightarrow\!\!\rightarrow \texttt{2\_abcc}\ldots, \left\{\begin{array}{l} \texttt{2caa\underline{b}}\ldots \\ \texttt{3caa\underline{b}}\ldots \end{array}\right\} \end{array} \right. \tag{159}$$

8 Generating the IRR

8.1 The list of results of the IRR(2)

The IRR(2) i.e. equation (4) were segregated into the 3 categories, those of type RR (160), type LL (161), which are non-extendable and are expressed unabbreviated and types RL and LR that are extendable and are abbreviated (162).

$$\begin{array}{llll} \texttt{1\underline{d}b} \rightarrow\!\!\rightarrow \texttt{3bc\_} & \texttt{1\underline{e}b} \rightarrow\!\!\rightarrow \texttt{3bc\_} & \texttt{2\underline{a}a} \rightarrow\!\!\rightarrow \texttt{2db\_} & \texttt{2\underline{a}b} \rightarrow\!\!\rightarrow \texttt{2db\_} \\ \texttt{2\underline{d}a} \rightarrow\!\!\rightarrow \texttt{2cb\_} & \texttt{3\underline{a}d} \rightarrow\!\!\rightarrow \texttt{2db\_} & \texttt{3\underline{e}b} \rightarrow\!\!\rightarrow \texttt{3bc\_} & \texttt{2\underline{c}a} \rightarrow\!\!\rightarrow \texttt{2ab\_} \\ \texttt{4\underline{e}a} \rightarrow\!\!\rightarrow \texttt{2cb\_} & \texttt{4\underline{e}b} \rightarrow\!\!\rightarrow \texttt{5ca\_} & \texttt{4\underline{c}b} \rightarrow\!\!\rightarrow \texttt{2ab\_} & \texttt{4\underline{c}d} \rightarrow\!\!\rightarrow \texttt{2db\_} \\ \texttt{5\underline{e}c} \rightarrow\!\!\rightarrow \texttt{2ab\_} & \texttt{4\underline{c}c} \rightarrow\!\!\rightarrow \texttt{2ab\_} & \texttt{5\underline{c}c} \rightarrow\!\!\rightarrow \texttt{2db\_} & \texttt{5\underline{e}c} \rightarrow\!\!\rightarrow \texttt{2db\_} \end{array} \tag{160}$$

$$\begin{array}{llll} \texttt{1a\underline{c}} \rightarrow\!\!\rightarrow \texttt{2\_ab} & \texttt{2e\underline{e}} \rightarrow\!\!\rightarrow \texttt{4\_ca} & \texttt{5b\underline{d}} \rightarrow\!\!\rightarrow \texttt{4\_cb} & \texttt{1b\underline{b}} \rightarrow\!\!\rightarrow \texttt{3\_ec} \\ \texttt{1e\underline{b}} \rightarrow\!\!\rightarrow \texttt{3\_ba} & & & \end{array} \tag{161}$$

$$
\begin{array}{llll}
\mathtt{1\underline{d}\ldots} \to\to \mathtt{4\_c\ldots} & \text{context } (\mathtt{e},\mathtt{c}) & \mathtt{1\underline{e}\ldots} \to\to \mathtt{4\_c\ldots} & \text{context } (\mathtt{e},\mathtt{c}) \\
\mathtt{2\underline{a}\ldots} \to\to \mathtt{2\_a\ldots} & \text{context } (\mathtt{c},\mathtt{a}) & \mathtt{2\underline{c}\ldots} \to\to \mathtt{5\_c\ldots} & \text{context } \begin{array}{c}(\mathtt{b},\mathtt{d})\\(\mathtt{c},\mathtt{a})\end{array} \\
\mathtt{2\underline{d}\ldots} \to\to \mathtt{2\_a\ldots} & \text{context } (\mathtt{c},\mathtt{b}) & \mathtt{2\underline{d}\ldots} \to\to \mathtt{3\_b\ldots} & \text{context } (\mathtt{b},\mathtt{d}) \\
\mathtt{3\underline{a}\ldots} \to\to \mathtt{2\_a\ldots} & \text{context } (\mathtt{a},\mathtt{b}) & \mathtt{3\underline{e}\ldots} \to\to \mathtt{3\_e\ldots} & \text{context } (\mathtt{d},\mathtt{c}) \\
\mathtt{3\underline{e}\ldots} \to\to \mathtt{4\_c\ldots} & \text{context } (\mathtt{c},\mathtt{a}) & \mathtt{4\underline{b}\ldots} \to\to \mathtt{3\_e\ldots} & \text{context } (\mathtt{d},\mathtt{c}) \\
\mathtt{4\underline{b}\ldots} \to\to \mathtt{4\_c\ldots} & \text{context } (\mathtt{a},\mathtt{b}) & \mathtt{4\underline{e}\ldots} \to\to \mathtt{3\_b\ldots} & \text{context } (\mathtt{d},\mathtt{a}) \\
\mathtt{5\underline{c}\ldots} \to\to \mathtt{2\_e\ldots} & \text{context } (\mathtt{d},\mathtt{c}) & \mathtt{5\underline{c}\ldots} \to\to \mathtt{3\_b\ldots} & \text{context } (\mathtt{b},\mathtt{c}) \\
\mathtt{5\underline{e}\ldots} \to\to \mathtt{2\_e\ldots} & \text{context } (\mathtt{d},\mathtt{c}) & \mathtt{5\underline{e}\ldots} \to\to \mathtt{3\_b\ldots} & \text{context } (\mathtt{b},\mathtt{c}) \\
\mathtt{1\ldots\underline{a}} \to\to \mathtt{2\ldots b\_} & \text{context } \begin{array}{c}(\mathtt{c},\mathtt{d})\\(\mathtt{a},\mathtt{c})\\(\mathtt{d},\mathtt{a})\end{array} & \mathtt{1\ldots\underline{b}} \to\to \mathtt{2\ldots b\_} & \text{context } (\mathtt{c},\mathtt{d}) \\
\mathtt{1\ldots\underline{c}} \to\to \mathtt{4\ldots c\_} & \text{context } (\mathtt{e},\mathtt{b}) & \mathtt{2\ldots\underline{b}} \to\to \mathtt{2\ldots b\_} & \text{context } (\mathtt{c},\mathtt{a}) \\
\mathtt{2\ldots\underline{b}} \to\to \mathtt{3\ldots a\_} & \text{context } (\mathtt{e},\mathtt{a}) & \mathtt{2\ldots\underline{b}} \to\to \mathtt{3\ldots c\_} & \text{context } (\mathtt{b},\mathtt{b}) \\
\mathtt{2\ldots\underline{e}} \to\to \mathtt{3\ldots c\_} & \text{context } (\mathtt{a},\mathtt{c}) & \mathtt{3\ldots\underline{b}} \to\to \mathtt{2\ldots b\_} & \text{context } (\mathtt{c},\mathtt{a}) \\
\mathtt{3\ldots\underline{b}} \to\to \mathtt{3\ldots a\_} & \text{context } (\mathtt{e},\mathtt{a}) & \mathtt{3\ldots\underline{b}} \to\to \mathtt{3\ldots c\_} & \text{context } (\mathtt{b},\mathtt{b}) \\
\mathtt{3\ldots\underline{c}} \to\to \mathtt{2\ldots b\_} & \text{context } \begin{array}{c}(\mathtt{c},\mathtt{a})\\(\mathtt{a},\mathtt{d})\\(\mathtt{d},\mathtt{c})\end{array} & \mathtt{3\ldots\underline{d}} \to\to \mathtt{2\ldots b\_} & \text{context } \begin{array}{c}(\mathtt{c},\mathtt{d})\\(\mathtt{e},\mathtt{d})\end{array} \\
\mathtt{4\ldots\underline{a}} \to\to \mathtt{3\ldots c\_} & \text{context } (\mathtt{e},\mathtt{b}) & \mathtt{4\ldots\underline{a}} \to\to \mathtt{4\ldots b\_} & \text{context } (\mathtt{a},\mathtt{c}) \\
\mathtt{4\ldots\underline{d}} \to\to \mathtt{2\ldots b\_} & \text{context } \begin{array}{c}(\mathtt{c},\mathtt{d})\\(\mathtt{e},\mathtt{d})\end{array} & \mathtt{5\ldots\underline{a}} \to\to \mathtt{2\ldots b\_} & \text{context } \begin{array}{c}(\mathtt{a},\mathtt{c})\\(\mathtt{c},\mathtt{d})\\(\mathtt{d},\mathtt{a})\end{array} \\
\mathtt{5\ldots\underline{b}} \to\to \mathtt{3\ldots b\_} & \text{context } (\mathtt{e},\mathtt{b}) & \mathtt{5\ldots\underline{b}} \to\to \mathtt{5\ldots a\_} & \text{context } (\mathtt{a},\mathtt{c}) \\
\mathtt{5\ldots\underline{d}} \to\to \mathtt{2\ldots b\_} & \text{context } (\mathtt{e},\mathtt{c}) & \mathtt{5\ldots\underline{d}} \to\to \mathtt{4\ldots c\_} & \text{context } (\mathtt{c},\mathtt{c})
\end{array}
\tag{162}
$$

8.2 Some results that generate the IRR($\mathtt{n} \geq 3$)

This is the complete set if $\mathtt{n} = 3$. The following are the results of applying $\mathtt{F}$ to (162)

$$
\mathtt{1\underline{d}\ldots} \to\to \mathtt{4\_c\ldots} \Rightarrow \left\{ \begin{array}{l} \mathtt{a} : \mathtt{2\underline{d}d\ldots} \to\to \mathtt{3\_bc\ldots} \\ \mathtt{c} : \mathtt{2\underline{a}d\ldots} \to\to \mathtt{2ab\ldots} \\ \mathtt{d} : \mathtt{2\underline{c}d\ldots} \to\to \mathtt{5\_cc\ldots} \end{array} \right. , \{\mathtt{1}\alpha\mathtt{\underline{d}\ldots}\}
\tag{163}
$$

$$
\mathtt{1\underline{e}\ldots} \to\to \mathtt{4\_c\ldots} \Rightarrow \left\{ \begin{array}{l} \mathtt{a} : \mathtt{2\underline{d}e\ldots} \to\to \mathtt{3\_bc\ldots} \\ \mathtt{c} : \mathtt{2\underline{a}e\ldots} \to\to \mathtt{2ab\ldots} \\ \mathtt{d} : \mathtt{2\underline{c}e\ldots} \to\to \mathtt{5\_cc\ldots} \end{array} \right. , \{\mathtt{1}\alpha\mathtt{\underline{e}\ldots}\}
\tag{164}
$$

$$
\mathtt{1}\underline{\beta}\mathtt{e\ldots} \to\to \mathtt{4\_cc\ldots} \Rightarrow \mathtt{c} : \mathtt{2\underline{a}}\beta\mathtt{e\ldots} \to\to \mathtt{1abd\ldots}, \emptyset \text{ for } \beta \in \{\mathtt{d},\mathtt{e}\}
\tag{165}
$$

$$\mathtt{2\underline{a}}\ldots \to\to \mathtt{2\_a}\ldots \Rightarrow \mathtt{b}:\ \left.\begin{array}{l}\mathtt{1\underline{d}a}\ldots\\ \mathtt{1\underline{e}a}\ldots\end{array}\right\} \to\to \mathtt{4\_ca}\ldots, \{2\alpha\mathtt{\underline{a}}\ldots\} \tag{166}$$

$$\mathtt{2\underline{a}c}\ldots \to\to \mathtt{2\_aa}\ldots \Rightarrow \emptyset, \emptyset \tag{167}$$

$$\mathtt{2\underline{c}}\ldots \to\to \mathtt{5\_c}\ldots \Rightarrow \mathtt{b}:\ \left.\begin{array}{l}\mathtt{1\underline{d}c}\ldots\\ \mathtt{1\underline{e}c}\ldots\end{array}\right\} \to\to \mathtt{3\_ec}\ldots, \{2\alpha\mathtt{\underline{c}}\ldots\} \tag{168}$$

$$\mathtt{2\underline{c}b}\ldots \to\to \mathtt{5\_cd}\ldots \Rightarrow \emptyset, \emptyset \tag{169}$$

$$\mathtt{2\underline{c}c}\ldots \to\to \mathtt{5\_ca}\ldots \Rightarrow \emptyset, \emptyset \tag{170}$$

$$\mathtt{2\underline{d}}\ldots \to\to \mathtt{2\_a}\ldots \Rightarrow \mathtt{b}:\ \left.\begin{array}{l}\mathtt{1\underline{d}d}\ldots\\ \mathtt{1\underline{e}d}\ldots\end{array}\right\} \to\to \mathtt{4\_ca}\ldots, \{2\alpha\mathtt{\underline{d}}\ldots\} \tag{171}$$

$$\mathtt{2\underline{d}c}\ldots \to\to \mathtt{2\_ab}\ldots \Rightarrow \emptyset, \emptyset \tag{172}$$

$$\mathtt{2\underline{d}}\ldots \to\to \mathtt{3\_b}\ldots \Rightarrow \mathtt{b}:\ \left.\begin{array}{l}\mathtt{1\underline{d}d}\ldots\\ \mathtt{1\underline{e}d}\ldots\end{array}\right\} \to\to \mathtt{4\_cb}\ldots, \{2\alpha\mathtt{\underline{d}}\ldots\} \tag{173}$$

$$\mathtt{2\underline{d}b}\ldots \to\to \mathtt{3\_bd}\ldots \Rightarrow \emptyset, \emptyset \tag{174}$$

$$\mathtt{3\underline{a}}\ldots \to\to \mathtt{2\_a}\ldots \Rightarrow \left\{\begin{array}{l}\mathtt{a}:\ \left.\begin{array}{l}\mathtt{5\underline{c}a}\ldots\\ \mathtt{5\underline{e}a}\ldots\end{array}\right\} \to\to \mathtt{2db}\ldots\\ \mathtt{b}: \mathtt{3\underline{e}a}\ldots \to\to \mathtt{4\_ca}\ldots\\ \mathtt{c}: \mathtt{4\underline{c}a}\ldots \to\to \mathtt{2ab}\ldots\end{array}\right. , \{3\alpha\mathtt{\underline{a}}\ldots\} \tag{175}$$

$$\mathtt{3\underline{a}a}\ldots \to\to \mathtt{2\_ab}\ldots \Rightarrow \left\{\begin{array}{l}\mathtt{a}:\ \left.\begin{array}{l}\mathtt{5\underline{c}aa}\ldots\\ \mathtt{5\underline{e}aa}\ldots\end{array}\right\} \to\to \mathtt{3dbc}\ldots\\ \mathtt{b}:\ \left.\begin{array}{l}\mathtt{1\underline{d}dc}\ldots\\ \mathtt{1\underline{e}dc}\ldots\end{array}\right\} \to\to \mathtt{4\_cab}\ldots\\ \mathtt{c}: \mathtt{4\underline{c}aa}\ldots \to\to \mathtt{3abc}\ldots\end{array}\right. , \{1\alpha\mathtt{a\underline{c}}\ldots\} \tag{176}$$

$$\mathtt{3\underline{e}}\ldots \to\to \mathtt{3\_e}\ldots \Rightarrow \left\{\begin{array}{l}\mathtt{a}:\ \left.\begin{array}{l}\mathtt{5\underline{c}e}\ldots\\ \mathtt{5\underline{e}e}\ldots\end{array}\right\} \to\to \mathtt{5ca}\ldots\\ \mathtt{b}: \mathtt{3\underline{e}e}\ldots \to\to \mathtt{4\_ce}\ldots\\ \mathtt{c}: \mathtt{4\underline{c}e}\ldots \to\to \mathtt{2\_ae}\ldots\end{array}\right. , \{3\alpha\mathtt{\underline{e}}\ldots\} \tag{177}$$

$$\mathtt{3\underline{e}d}\ldots \to\to \mathtt{3\_ec} \Rightarrow \mathtt{a}:\ \left.\begin{array}{l}\mathtt{5\underline{c}ed}\ldots\\ \mathtt{5\underline{e}ed}\ldots\end{array}\right\} \to\to \mathtt{3caa}\ldots, \emptyset \tag{178}$$

$$\mathtt{3\underline{e}}\ldots \rightarrow\rightarrow \mathtt{4\_c}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{a}: \left.\begin{array}{l}\mathtt{5\underline{c}e}\ldots\\ \mathtt{5\underline{e}e}\ldots\end{array}\right\} \rightarrow\rightarrow \mathtt{3\_bc}\ldots \\ \mathtt{b}: \mathtt{3\underline{e}e}\ldots \rightarrow\rightarrow \mathtt{3bc}\ldots \\ \mathtt{c}: \mathtt{4\underline{c}e}\ldots \rightarrow\rightarrow \mathtt{2ab}\ldots \end{array}\right. , \{\mathtt{3}\alpha\mathtt{\underline{e}}\ldots\} \tag{179}$$

$$\mathtt{3\underline{e}c}\ldots \rightarrow\rightarrow \mathtt{4\_ca} \Rightarrow \left\{\begin{array}{l} \mathtt{b}: \mathtt{3\underline{e}ec}\ldots \rightarrow\rightarrow \mathtt{4bcc}\ldots \\ \mathtt{c}: \mathtt{4\underline{c}ec}\ldots \rightarrow\rightarrow \mathtt{1abc}\ldots \end{array}\right. , \{\mathtt{2}\alpha\mathtt{e\underline{e}}\ldots\} \tag{180}$$

$$\mathtt{4\underline{b}}\ldots \rightarrow\rightarrow \mathtt{3\_e}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{b}: \mathtt{4\underline{b}b}\ldots \rightarrow\rightarrow \mathtt{4\_ce}\ldots \\ \mathtt{c}: \mathtt{3\underline{a}b}\ldots \rightarrow\rightarrow \mathtt{2\_ae}\ldots \end{array}\right. , \{\mathtt{4}\alpha\mathtt{\underline{b}}\ldots\} \tag{181}$$

$$\mathtt{4\underline{b}d}\ldots \rightarrow\rightarrow \mathtt{3\_ec}\ldots \Rightarrow \emptyset, \{\mathtt{1}\alpha\mathtt{b\underline{b}}\ldots\} \tag{182}$$

$$\mathtt{4\underline{b}}\ldots \rightarrow\rightarrow \mathtt{4\_c}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{b}: \mathtt{4\underline{b}b}\ldots \rightarrow\rightarrow \mathtt{3bc}\ldots \\ \mathtt{c}: \mathtt{3\underline{a}b}\ldots \rightarrow\rightarrow \mathtt{2ab}\ldots \end{array}\right. , \{\mathtt{4}\alpha\mathtt{\underline{b}}\ldots\} \tag{183}$$

$$\mathtt{4\underline{b}a}\ldots \rightarrow\rightarrow \mathtt{4\_cb}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{b}: \mathtt{4\underline{b}ba}\ldots \rightarrow\rightarrow \mathtt{2bab}\ldots \\ \mathtt{c}: \mathtt{3\underline{a}ba}\ldots \rightarrow\rightarrow \mathtt{3abc}\ldots \end{array}\right. , \{\mathtt{5}\alpha\mathtt{b\underline{d}}\ldots\} \tag{184}$$

$$\mathtt{4\underline{e}}\ldots \rightarrow\rightarrow \mathtt{3\_b}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{b}: \mathtt{4\underline{b}e}\ldots \rightarrow\rightarrow \mathtt{4\_cb}\ldots \\ \mathtt{c}: \mathtt{3\underline{a}e}\ldots \rightarrow\rightarrow \mathtt{2\_ab}\ldots \end{array}\right. , \{\mathtt{4}\alpha\mathtt{\underline{e}}\ldots\} \tag{185}$$

$$\mathtt{4\underline{e}d}\ldots \rightarrow\rightarrow \mathtt{3\_ba}\ldots \Rightarrow \emptyset, \{\mathtt{1}\alpha\mathtt{e\underline{b}}\ldots\} \tag{186}$$

$$\mathtt{5}\underline{\beta}\ldots \rightarrow\rightarrow \mathtt{2\_e}\ldots \Rightarrow \mathtt{a}: \mathtt{4\underline{e}}\beta\ldots \rightarrow\rightarrow \mathtt{2cb}\ldots, \{\mathtt{5}\alpha\underline{\beta}\ldots\} \text{ for } \beta \in \{\mathtt{c},\mathtt{e}\} \tag{187}$$

$$\mathtt{5}\underline{\beta}\mathtt{d}\ldots \rightarrow\rightarrow \mathtt{2\_ec}\ldots \Rightarrow \mathtt{a}: \mathtt{4\underline{e}}\beta\mathtt{d}\ldots \rightarrow\rightarrow \mathtt{1cbd}\ldots, \emptyset \text{ for } \beta \in \{\mathtt{c},\mathtt{e}\} \tag{188}$$

$$\mathtt{5}\underline{\beta}\ldots \rightarrow\rightarrow \mathtt{3\_b}\ldots \Rightarrow \mathtt{a}: \mathtt{4\underline{e}}\beta\ldots \rightarrow\rightarrow \mathtt{4cb}\ldots, \{\mathtt{5}\alpha\underline{\beta}\ldots\} \text{ for } \beta \in \{\mathtt{c},\mathtt{e}\} \tag{189}$$

$$\mathtt{5}\underline{\beta}\mathtt{b}\ldots \rightarrow\rightarrow \mathtt{3\_bc}\ldots \Rightarrow \mathtt{a}: \mathtt{4\underline{e}}\beta\mathtt{b}\ldots \rightarrow\rightarrow \mathtt{3cbc}\ldots, \emptyset \text{ for } \beta \in \{\mathtt{c},\mathtt{e}\} \tag{190}$$

$$\mathtt{1}\ldots\mathtt{\underline{a}} \rightarrow\rightarrow \mathtt{2}\ldots\mathtt{b\_} \Rightarrow \emptyset, \{\mathtt{1}\ldots\mathtt{\underline{a}}\alpha\} \tag{191}$$

$$\mathtt{1}\ldots\mathtt{\underline{b}} \rightarrow\rightarrow \mathtt{2}\ldots\mathtt{b\_} \Rightarrow \emptyset, \{\mathtt{1}\ldots\mathtt{\underline{b}}\alpha\} \tag{192}$$

$$\mathtt{1}\ldots\underline{\mathtt{c}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{c\_} \Rightarrow \emptyset, \{\mathtt{1}\ldots\underline{\mathtt{c}}\alpha\} \tag{193}$$

$$\mathtt{2}\ldots\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{b\_} \Rightarrow \left\{\begin{array}{l} \mathtt{a}:\mathtt{3}\ldots\mathtt{b}\underline{\mathtt{c}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bc\_} \\ \mathtt{d}:\mathtt{1}\ldots\mathtt{b}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{ba\_} \\ \mathtt{e}:\mathtt{5}\ldots\mathtt{b}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{cc} \end{array}\right., \{\mathtt{2}\ldots\underline{\mathtt{b}}\alpha\} \tag{194}$$

$$\mathtt{2}\ldots\mathtt{c}\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab\_} \Rightarrow \mathtt{e}:\mathtt{5}\ldots\mathtt{cb}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bcc}, \emptyset \tag{195}$$

$$\mathtt{2}\ldots\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{a\_} \Rightarrow \left\{\begin{array}{l} \mathtt{a}:\mathtt{3}\ldots\mathtt{b}\underline{\mathtt{c}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{ac\_} \\ \mathtt{d}:\mathtt{1}\ldots\mathtt{b}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{ec} \\ \mathtt{e}:\mathtt{5}\ldots\mathtt{b}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{ab\_} \end{array}\right., \{\mathtt{2}\ldots\underline{\mathtt{b}}\alpha\} \tag{196}$$

$$\mathtt{2}\ldots\mathtt{e}\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{aa\_} \Rightarrow \mathtt{d}:\mathtt{1}\ldots\mathtt{eb}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{cbd\_}, \emptyset \tag{197}$$

$$\mathtt{2}\ldots\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{c\_} \Rightarrow \left\{\begin{array}{l} \mathtt{a}:\mathtt{3}\ldots\mathtt{b}\underline{\mathtt{c}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{cc\_} \\ \mathtt{d}:\mathtt{1}\ldots\mathtt{b}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{db\_} \\ \mathtt{e}:\mathtt{5}\ldots\mathtt{b}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{cb\_} \end{array}\right., \{\mathtt{2}\ldots\underline{\mathtt{b}}\alpha\} \tag{198}$$

$$\mathtt{2}\ldots\mathtt{b}\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bc\_} \Rightarrow \emptyset, \{\mathtt{1}\ldots\underline{\mathtt{d}}\mathtt{b}\alpha, \mathtt{1}\ldots\underline{\mathtt{e}}\mathtt{b}\alpha\} \tag{199}$$

$$\mathtt{2}\ldots\underline{\mathtt{e}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{c\_} \Rightarrow \left\{\begin{array}{l} \mathtt{a}:\mathtt{3}\ldots\mathtt{e}\underline{\mathtt{c}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{cc\_} \\ \mathtt{d}:\mathtt{1}\ldots\mathtt{e}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{db\_} \\ \mathtt{e}:\mathtt{5}\ldots\mathtt{e}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{cb\_} \end{array}\right., \{\mathtt{2}\ldots\underline{\mathtt{e}}\alpha\} \tag{200}$$

$$\mathtt{2}\ldots\mathtt{a}\underline{\mathtt{e}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{cc\_} \Rightarrow \emptyset, \emptyset \tag{201}$$

$$\mathtt{3}\ldots\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{a\_} \Rightarrow \left\{\begin{array}{l} \mathtt{a}:\mathtt{1}\ldots\mathtt{b}\underline{\mathtt{c}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{ac\_} \\ \mathtt{b}:\mathtt{4}\ldots\mathtt{b}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bc} \\ \mathtt{c}:\mathtt{2}\ldots\mathtt{b}\underline{\mathtt{e}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{db\_} \\ \mathtt{e}:\mathtt{5}\ldots\mathtt{b}\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{ab\_} \end{array}\right., \{\mathtt{3}\ldots\underline{\mathtt{b}}\alpha\} \tag{202}$$

$$\mathtt{3}\ldots\mathtt{e}\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{aa\_} \Rightarrow \mathtt{b}:\mathtt{4}\ldots\mathtt{eb}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{cbc\_}, \emptyset \tag{203}$$

$$\mathtt{3}\ldots\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{c\_} \Rightarrow \left\{\begin{array}{l} \mathtt{a}:\mathtt{1}\ldots\mathtt{b}\underline{\mathtt{c}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{cc\_} \\ \mathtt{b}:\mathtt{4}\ldots\mathtt{b}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab\_} \\ \mathtt{c}:\mathtt{2}\ldots\mathtt{b}\underline{\mathtt{e}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab\_} \\ \mathtt{e}:\mathtt{5}\ldots\mathtt{b}\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{cb\_} \end{array}\right., \{\mathtt{3}\ldots\underline{\mathtt{b}}\alpha\} \tag{204}$$

$$\mathtt{3}\ldots\mathtt{b\underline{b}} \to\to \mathtt{3}\ldots\mathtt{bc\_} \Rightarrow \left\{\begin{array}{l} \mathtt{a}: \mathtt{5}\ldots\mathtt{ea\underline{d}} \to\to \mathtt{4}\ldots\mathtt{bcc\_} \\ \mathtt{c}: \left.\begin{array}{c} \mathtt{2}\ldots\mathtt{ea\underline{b}} \\ \mathtt{3}\ldots\mathtt{ea\underline{b}} \end{array}\right\} \to\to \mathtt{2}\ldots\mathtt{bab\_} \\ \mathtt{d}: \mathtt{1}\ldots\mathtt{ea\underline{b}} \to\to \mathtt{2}\ldots\mathtt{bdb\_} \end{array}\right. ,\{\mathtt{3}\ldots\mathtt{\underline{e}b}\alpha\} \tag{205}$$

$$\mathtt{3}\ldots\underline{\beta} \to\to \mathtt{2}\ldots\mathtt{b\_} \Rightarrow \left\{\begin{array}{l} \mathtt{a}: \mathtt{1}\ldots\beta\mathtt{\underline{c}} \to\to \mathtt{1}\ldots\mathtt{bc\_} \\ \mathtt{b}: \mathtt{4}\ldots\beta\mathtt{\underline{a}} \to\to \mathtt{3}\ldots\mathtt{bc\_} \\ \mathtt{c}: \mathtt{2}\ldots\beta\mathtt{\underline{e}} \to\to \mathtt{1}\ldots\mathtt{bd\_} \\ \mathtt{e}: \mathtt{5}\ldots\beta\mathtt{\underline{b}} \to\to \mathtt{4}\ldots\mathtt{cc} \end{array}\right. ,\{\mathtt{3}\ldots\underline{\beta}\alpha\} \text{ for } \beta \in \{\mathtt{b},\mathtt{c},\mathtt{d}\} \tag{206}$$

$$\mathtt{3}\ldots\mathtt{c\underline{b}} \to\to \mathtt{2}\ldots\mathtt{ab\_} \Rightarrow \mathtt{e}: \mathtt{5}\ldots\mathtt{cb\underline{b}} \to\to \mathtt{3}\ldots\mathtt{bcc}, \{\mathtt{4}\ldots\mathtt{\underline{c}b}\alpha\} \tag{207}$$

$$\mathtt{3}\ldots\mathtt{a\underline{c}} \to\to \mathtt{2}\ldots\mathtt{db\_} \Rightarrow \left\{\begin{array}{ll} \mathtt{a}: & \left.\begin{array}{c} \mathtt{1}\ldots\mathtt{cd\underline{c}} \\ \mathtt{5}\ldots\mathtt{cd\underline{d}} \\ \mathtt{1}\ldots\mathtt{ed\underline{c}} \\ \mathtt{5}\ldots\mathtt{ed\underline{d}} \end{array}\right\} \to\to \mathtt{1}\ldots\mathtt{dbc\_} \\ \mathtt{b}: & \left.\begin{array}{c} \mathtt{4}\ldots\mathtt{cd\underline{a}} \\ \mathtt{4}\ldots\mathtt{ed\underline{a}} \end{array}\right\} \to\to \mathtt{3}\ldots\mathtt{dbc\_} \\ \mathtt{c}: & \left.\begin{array}{c} \mathtt{2}\ldots\mathtt{cd\underline{e}} \\ \mathtt{2}\ldots\mathtt{cd\underline{b}} \\ \mathtt{3}\ldots\mathtt{cd\underline{b}} \\ \mathtt{2}\ldots\mathtt{ed\underline{e}} \\ \mathtt{2}\ldots\mathtt{ed\underline{b}} \\ \mathtt{3}\ldots\mathtt{ed\underline{b}} \end{array}\right\} \to\to \mathtt{1}\ldots\mathtt{dbd\_} \\ \mathtt{d}: & \left.\begin{array}{c} \mathtt{1}\ldots\mathtt{cd\underline{b}} \\ \mathtt{1}\ldots\mathtt{ed\underline{b}} \end{array}\right\} \to\to \mathtt{1}\ldots\mathtt{dba\_} \\ \mathtt{e}: & \left.\begin{array}{c} \mathtt{5}\ldots\mathtt{cd\underline{b}} \\ \mathtt{5}\ldots\mathtt{ed\underline{b}} \end{array}\right\} \to\to \mathtt{5}\ldots\mathtt{ccc} \end{array}\right. \tag{208}$$

$$\mathtt{3}\ldots\mathtt{d\underline{c}} \to\to \mathtt{2}\ldots\mathtt{cb\_} \Rightarrow \mathtt{e}: \mathtt{5}\ldots\mathtt{dc\underline{b}} \to\to \mathtt{1}\ldots\mathtt{abd\_}, \emptyset \tag{209}$$

$$\mathtt{3}\ldots\mathtt{c\underline{d}} \to\to \mathtt{2}\ldots\mathtt{db\_} \Rightarrow \mathtt{e}: \mathtt{5}\ldots\mathtt{cd\underline{b}} \to\to \mathtt{5}\ldots\mathtt{ccc}, \{\mathtt{4}\ldots\mathtt{\underline{c}d}\alpha, \mathtt{2}\ldots\mathtt{\underline{a}b}\alpha\} \tag{210}$$

$$\mathtt{3}\ldots\mathtt{e\underline{d}} \to\to \mathtt{2}\ldots\mathtt{db\_} \Rightarrow \mathtt{e}: \mathtt{5}\ldots\mathtt{ed\underline{b}} \to\to \mathtt{5}\ldots\mathtt{ccc}, \{\mathtt{4}\ldots\mathtt{\underline{c}d}\alpha, \mathtt{2}\ldots\mathtt{\underline{a}b}\alpha\} \tag{211}$$

$$4\ldots\underline{\mathtt{a}} \to\!\!\to 3\ldots\mathtt{c\_} \Rightarrow \begin{cases} \mathtt{a}: 5\ldots\mathtt{a}\underline{\mathtt{d}} \to\!\!\to 4\ldots\mathtt{cc\_} \\ \mathtt{c}: \left.\begin{array}{l} 2\ldots\mathtt{a}\underline{\mathtt{b}} \\ 3\ldots\mathtt{a}\underline{\mathtt{b}} \end{array}\right\} \to\!\!\to 2\ldots\mathtt{ab\_} \\ \mathtt{d}: 1\ldots\mathtt{a}\underline{\mathtt{b}} \to\!\!\to 2\ldots\mathtt{db\_} \end{cases} , \{4\ldots\underline{\mathtt{a}}\alpha\} \tag{212}$$

$$4\ldots\mathtt{e}\underline{\mathtt{a}} \to\!\!\to 3\ldots\mathtt{bc\_} \Rightarrow \emptyset, \emptyset \tag{213}$$

$$4\ldots\underline{\mathtt{a}} \to\!\!\to 4\ldots\mathtt{b\_} \Rightarrow \begin{cases} \mathtt{a}: 5\ldots\mathtt{a}\underline{\mathtt{d}} \to\!\!\to 4\ldots\mathtt{cb} \\ \mathtt{c}: \left.\begin{array}{l} 2\ldots\mathtt{a}\underline{\mathtt{b}} \\ 3\ldots\mathtt{a}\underline{\mathtt{b}} \end{array}\right\} \to\!\!\to 3\ldots\mathtt{bc\_} \\ \mathtt{d}: 1\ldots\mathtt{a}\underline{\mathtt{b}} \to\!\!\to 3\ldots\mathtt{ec} \end{cases} , \{4\ldots\underline{\mathtt{a}}\alpha\} \tag{214}$$

$$4\ldots\mathtt{a}\underline{\mathtt{a}} \to\!\!\to 4\ldots\mathtt{cb\_} \Rightarrow \begin{cases} \mathtt{a}: 5\ldots\mathtt{aa}\underline{\mathtt{d}} \to\!\!\to 3\ldots\mathtt{abc\_} \\ \mathtt{d}: 1\ldots\mathtt{aa}\underline{\mathtt{b}} \to\!\!\to 2\ldots\mathtt{aec} \end{cases} , \emptyset \tag{215}$$

$$4\ldots\underline{\mathtt{d}} \to\!\!\to 2\ldots\mathtt{b\_} \Rightarrow \begin{cases} \mathtt{a}: 5\ldots\mathtt{d}\underline{\mathtt{d}} \to\!\!\to 1\ldots\mathtt{bc\_} \\ \mathtt{c}: \left.\begin{array}{l} 2\ldots\mathtt{d}\underline{\mathtt{b}} \\ 3\ldots\mathtt{d}\underline{\mathtt{b}} \end{array}\right\} \to\!\!\to 1\ldots\mathtt{bd\_} \\ \mathtt{d}: 1\ldots\mathtt{d}\underline{\mathtt{b}} \to\!\!\to 1\ldots\mathtt{ba\_} \end{cases} , \{4\ldots\underline{\mathtt{d}}\alpha\} \tag{216}$$

$$4\ldots\mathtt{c}\underline{\mathtt{d}} \to\!\!\to 2\ldots\mathtt{db\_} \Rightarrow \emptyset, \{3\ldots\underline{\mathtt{a}}\mathtt{d}\alpha\} \tag{217}$$

$$4\ldots\mathtt{e}\underline{\mathtt{d}} \to\!\!\to 2\ldots\mathtt{db\_} \Rightarrow \emptyset, \emptyset \tag{218}$$

$$5\ldots\underline{\beta} \to\!\!\to 2\ldots\mathtt{b\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l} 3\ldots\beta\underline{\mathtt{d}} \\ 4\ldots\beta\underline{\mathtt{d}} \end{array}\right\} \to\!\!\to 1\ldots\mathtt{bd\_}, \{5\ldots\underline{\beta}\alpha\} \text{ for } \beta \in \{\mathtt{a},\mathtt{d}\} \tag{219}$$

$$5\ldots\mathtt{a}\underline{\mathtt{a}} \to\!\!\to 2\ldots\mathtt{cb\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l} 3\ldots\mathtt{ed}\underline{\mathtt{d}} \\ 4\ldots\mathtt{ed}\underline{\mathtt{d}} \end{array}\right\} \to\!\!\to 1\ldots\mathtt{cbd\_}, \{4\ldots\underline{\mathtt{e}}\mathtt{a}\alpha\} \tag{220}$$

$$5\ldots\mathtt{c}\underline{\mathtt{a}} \to\!\!\to 2\ldots\mathtt{db\_} \Rightarrow \emptyset, \emptyset \tag{221}$$

$$5\ldots\mathtt{d}\underline{\mathtt{a}} \to\!\!\to 2\ldots\mathtt{ab\_} \Rightarrow \emptyset, \emptyset \tag{222}$$

$$5\ldots\mathtt{e}\underline{\mathtt{d}} \to\!\!\to 2\ldots\mathtt{cb\_} \Rightarrow \emptyset, \emptyset \tag{223}$$

$$\texttt{5}\ldots\underline{\texttt{b}} \to\to \texttt{3}\ldots\texttt{b\_} \Rightarrow \texttt{c}: \left.\begin{array}{l}\texttt{3}\ldots\texttt{b}\underline{\texttt{d}}\\ \texttt{4}\ldots\texttt{b}\underline{\texttt{d}}\end{array}\right\} \to\to \texttt{4}\ldots\texttt{ca}, \{\texttt{5}\ldots\underline{\texttt{b}}\alpha\} \tag{224}$$

$$\texttt{5}\ldots\texttt{e}\underline{\texttt{b}} \to\to \texttt{3}\ldots\texttt{bb\_} \Rightarrow \texttt{c}: \left.\begin{array}{l}\texttt{3}\ldots\texttt{eb}\underline{\texttt{d}}\\ \texttt{4}\ldots\texttt{eb}\underline{\texttt{d}}\end{array}\right\} \to\to \texttt{4}\ldots\texttt{bcc\_}, \emptyset \tag{225}$$

$$\texttt{5}\ldots\underline{\texttt{b}} \to\to \texttt{5}\ldots\texttt{a\_} \Rightarrow \texttt{c}: \left.\begin{array}{l}\texttt{3}\ldots\texttt{b}\underline{\texttt{d}}\\ \texttt{4}\ldots\texttt{b}\underline{\texttt{d}}\end{array}\right\} \to\to \texttt{3}\ldots\texttt{aa\_}, \{\texttt{5}\ldots\underline{\texttt{b}}\alpha\} \tag{226}$$

$$\texttt{5}\ldots\texttt{a}\underline{\texttt{b}} \to\to \texttt{5}\ldots\texttt{ca\_} \Rightarrow \emptyset, \{\texttt{4}\ldots\underline{\texttt{e}}\texttt{b}\alpha\} \tag{227}$$

$$\texttt{5}\ldots\underline{\texttt{d}} \to\to \texttt{4}\ldots\texttt{c\_} \Rightarrow \texttt{c}: \left.\begin{array}{l}\texttt{3}\ldots\texttt{d}\underline{\texttt{d}}\\ \texttt{4}\ldots\texttt{d}\underline{\texttt{d}}\end{array}\right\} \to\to \texttt{3}\ldots\texttt{cc\_}, \{\texttt{5}\ldots\underline{\texttt{d}}\alpha\} \tag{228}$$

$$\texttt{5}\ldots\texttt{c}\underline{\texttt{d}} \to\to \texttt{4}\ldots\texttt{cc\_} \Rightarrow \emptyset, \emptyset \tag{229}$$

The right hand members of the results in (163)-(229) represent all the IRR(3) and their derivations i.e. if the IGR's in this set that have length 3 are applied to the members of IRR(2), all the members of IRR(3) can be generated and no results in this list can be omitted for this purpose. For the next step it must be verified that all the members of IRR(4) can be generated from the IRR(3) using appropriate IGR's which will be found. To to this `F` needs to be applied to all members of the set (163)-(229) that have length 3. The others are their derivations one symbol at a time using the method described in Section 5.

8.3 Additional results that generate the IRR(4)

These are the results of applying `F` to the RHS's of the IGR's that generate all the IRR(3).

$$\texttt{1}\underline{\gamma}\ldots \to\to \texttt{3\_e}\ldots \Rightarrow \left\{\begin{array}{l}\texttt{a}: \texttt{2}\underline{\texttt{d}}\gamma\ldots \to\to \texttt{5ca}\ldots\\ \texttt{c}: \texttt{2}\underline{\texttt{a}}\gamma\ldots \to\to \texttt{2\_ae}\ldots\\ \texttt{d}: \texttt{2}\underline{\texttt{c}}\gamma\ldots \to\to \texttt{5\_ce}\ldots\end{array}\right. \quad \text{for } \gamma \in \{\texttt{d}, \texttt{e}\} \tag{230}$$

$$\texttt{1}\underline{\beta}\texttt{a}\ldots \to\to \texttt{4\_ca}\ldots \Rightarrow \texttt{c}: \texttt{2}\underline{\texttt{a}}\beta\texttt{a}\ldots \to\to \texttt{1abc}\ldots, \emptyset \text{ for } \beta \in \{\texttt{d}, \texttt{e}\} \tag{231}$$

$$\texttt{1}\underline{\beta}\texttt{ac}\ldots \to\to \texttt{4\_caa}\ldots \Rightarrow \texttt{c}: \texttt{2}\underline{\texttt{a}}\beta\texttt{ac}\ldots \to\to \texttt{2abdb}\ldots, \emptyset \text{ for } \beta \in \{\texttt{d}, \texttt{e}\} \tag{232}$$

$$\texttt{1}\underline{\beta}\texttt{c}\ldots \to\to \texttt{3\_ec}\ldots \Rightarrow \texttt{a}: \texttt{2}\underline{\texttt{d}}\beta\texttt{c}\ldots \to\to \texttt{3caa}\ldots, \emptyset \text{ for } \beta \in \{\texttt{d}, \texttt{e}\} \tag{233}$$

$$\mathtt{1}\underline{\beta}\mathtt{cb}\ldots \rightarrow\rightarrow \mathtt{3\_ecd}\ldots \Rightarrow \mathtt{a} : \mathtt{2}\underline{\mathtt{d}}\beta\mathtt{cb}\ldots \rightarrow\rightarrow \mathtt{1ccbd}\ldots, \emptyset \text{ for } \beta \in \{\mathtt{d}, \mathtt{e}\} \tag{234}$$

$$\mathtt{1}\underline{\beta}\mathtt{cc}\ldots \rightarrow\rightarrow \mathtt{3\_eca}\ldots \Rightarrow \mathtt{a} : \mathtt{2}\underline{\mathtt{d}}\beta\mathtt{cc}\ldots \rightarrow\rightarrow \mathtt{4caac}\ldots, \emptyset \text{ for } \beta \in \{\mathtt{d}, \mathtt{e}\} \tag{235}$$

$$\mathtt{1}\underline{\beta}\mathtt{d}\ldots \rightarrow\rightarrow \mathtt{4\_ca}\ldots \Rightarrow \mathtt{c} : \mathtt{2}\underline{\mathtt{a}}\beta\mathtt{d}\ldots \rightarrow\rightarrow \mathtt{1abc}\ldots, \emptyset \text{ for } \beta \in \{\mathtt{d}, \mathtt{e}\} \tag{236}$$

$$\mathtt{1}\underline{\beta}\mathtt{dc}\ldots \rightarrow\rightarrow \mathtt{4\_cab}\ldots \Rightarrow \mathtt{c} : \mathtt{2}\underline{\mathtt{a}}\beta\mathtt{dc}\ldots \rightarrow\rightarrow \mathtt{2abdb}\ldots, \emptyset \text{ for } \beta \in \{\mathtt{d}, \mathtt{e}\} \tag{237}$$

$$\mathtt{1}\underline{\beta}\mathtt{d}\ldots \rightarrow\rightarrow \mathtt{4\_cb}\ldots \Rightarrow \mathtt{c} : \mathtt{2}\underline{\mathtt{a}}\beta\mathtt{d}\ldots \rightarrow\rightarrow \mathtt{3abc}\ldots, \emptyset \text{ for } \beta \in \{\mathtt{d}, \mathtt{e}\} \tag{238}$$

$$\mathtt{1}\underline{\beta}\mathtt{db}\ldots \rightarrow\rightarrow \mathtt{4\_cbd}\ldots \Rightarrow \mathtt{c} : \mathtt{2}\underline{\mathtt{a}}\beta\mathtt{db}\ldots \rightarrow\rightarrow \mathtt{2abdb}\ldots, \emptyset \text{ for } \beta \in \{\mathtt{d}, \mathtt{e}\} \tag{239}$$

$$\mathtt{2}\underline{\mathtt{c}}\mathtt{d}\ldots \rightarrow\rightarrow \mathtt{5\_cc}\ldots \Rightarrow \left\{ \begin{array}{ll} \mathtt{a} : & \left.\begin{array}{l} \mathtt{4}\underline{\mathtt{e}}\mathtt{cc}\ldots \\ \mathtt{4}\underline{\mathtt{e}}\mathtt{ec}\ldots \\ \mathtt{5}\underline{\mathtt{c}}\mathtt{ad}\ldots \\ \mathtt{5}\underline{\mathtt{e}}\mathtt{ad}\ldots \end{array}\right\} \rightarrow\rightarrow \mathtt{2\_ecc}\ldots \\ \mathtt{b} : & \left.\begin{array}{l} \mathtt{1}\underline{\mathtt{d}}\mathtt{aa}\ldots \\ \mathtt{1}\underline{\mathtt{e}}\mathtt{aa}\ldots \\ \mathtt{1}\underline{\mathtt{d}}\mathtt{ab}\ldots \\ \mathtt{1}\underline{\mathtt{e}}\mathtt{ab}\ldots \\ \mathtt{4}\underline{\mathtt{b}}\mathtt{cd}\ldots \\ \mathtt{3}\underline{\mathtt{e}}\mathtt{ad}\ldots \end{array}\right\} \rightarrow\rightarrow \mathtt{3\_ecc}\ldots \\ \mathtt{c} : & \left.\begin{array}{l} \mathtt{3}\underline{\mathtt{a}}\mathtt{cd}\ldots \\ \mathtt{4}\underline{\mathtt{c}}\mathtt{ad}\ldots \end{array}\right\} \rightarrow\rightarrow \mathtt{1dbd}\ldots \end{array}\right. , \left\{ \begin{array}{l} \mathtt{1}\alpha\mathtt{c}\underline{\mathtt{a}}\ldots \\ \mathtt{1}\alpha\mathtt{c}\underline{\mathtt{b}}\ldots \\ \mathtt{3}\alpha\mathtt{a}\underline{\mathtt{c}}\ldots \\ \mathtt{3}\alpha\mathtt{c}\underline{\mathtt{d}}\ldots \\ \mathtt{4}\alpha\mathtt{c}\underline{\mathtt{d}}\ldots \\ \mathtt{3}\alpha\mathtt{e}\underline{\mathtt{d}}\ldots \\ \mathtt{4}\alpha\mathtt{e}\underline{\mathtt{d}}\ldots \end{array}\right\} \tag{240}$$

$$\mathtt{2}\underline{\mathtt{c}}\mathtt{de}\ldots \rightarrow\rightarrow \mathtt{5\_ccc}\ldots \Rightarrow \mathtt{c} : \left.\begin{array}{l} \mathtt{3}\underline{\mathtt{a}}\mathtt{cde}\ldots \\ \mathtt{4}\underline{\mathtt{c}}\mathtt{ade}\ldots \end{array}\right\} \rightarrow\rightarrow \mathtt{5\_ceca}\ldots, \left\{ \begin{array}{l} \mathtt{5cac}\underline{\mathtt{b}}\ldots \\ \mathtt{5ccd}\underline{\mathtt{b}}\ldots \\ \mathtt{5ced}\underline{\mathtt{b}}\ldots \end{array}\right\} \tag{241}$$

$$\mathtt{2}\underline{\mathtt{c}}\mathtt{e}\ldots \rightarrow\rightarrow \mathtt{5\_cc}\ldots \Rightarrow \emptyset, \{\mathtt{5}\alpha\mathtt{c}\underline{\mathtt{a}}\ldots\} \tag{242}$$

$$\mathtt{2\underline{d}d}\ldots \to\to \mathtt{3\_bc}\ldots \Rightarrow \left\{ \begin{array}{ll} \mathtt{b:} & \left.\begin{array}{l} \mathtt{1\underline{d}ca}\ldots \\ \mathtt{1\underline{e}ca}\ldots \\ \mathtt{4\underline{b}cc}\ldots \\ \mathtt{4\underline{b}cb}\ldots \end{array}\right\} \to\to \mathtt{4\_cbc}\ldots \\ \mathtt{c:} & \left.\begin{array}{l} \mathtt{3\underline{a}cc}\ldots \\ \mathtt{3\underline{a}cb}\ldots \end{array}\right\} \to\to \mathtt{2\_abc}\ldots \end{array}\right. , \left\{ \begin{array}{l} \mathtt{3}\alpha\mathtt{c\underline{c}}\ldots \\ \mathtt{1}\alpha\mathtt{d\underline{a}}\ldots \\ \mathtt{2}\alpha\mathtt{c\underline{b}}\ldots \end{array}\right\} \tag{243}$$

$$\mathtt{2\underline{d}de}\ldots \to\to \mathtt{3\_bcc}\ldots \Rightarrow \emptyset, \emptyset \tag{244}$$

$$\mathtt{2\underline{d}e}\ldots \to\to \mathtt{3\_bc}\ldots \Rightarrow \emptyset, \{\mathtt{5}\alpha\mathtt{d\underline{a}}\ldots\} \tag{245}$$

$$\mathtt{3\underline{a}b}\ldots \to\to \mathtt{2\_ae}\ldots \Rightarrow \left\{ \begin{array}{l} \mathtt{a:} \left.\begin{array}{l} \mathtt{5\underline{c}ab}\ldots \\ \mathtt{5\underline{e}ab}\ldots \end{array}\right\} \to\to \mathtt{5\_ccc}\ldots \\ \mathtt{c:4\underline{c}ab}\ldots \to\to \mathtt{3\_bcc}\ldots \end{array}\right. , \{\mathtt{4}\alpha\mathtt{a\underline{a}}\ldots\} \tag{246}$$

$$\mathtt{3\underline{a}bd}\ldots \to\to \mathtt{2\_aec}\ldots \Rightarrow \left\{ \begin{array}{ll} \mathtt{b:} & \left.\begin{array}{l} \mathtt{4\underline{b}ecb}\ldots \\ \mathtt{4\underline{b}eeb}\ldots \end{array}\right\} \to\to \mathtt{4\_caec}\ldots, \left\{\begin{array}{l} \mathtt{2baa\underline{b}}\ldots \\ \mathtt{3baa\underline{b}}\ldots \end{array}\right\} \\ \mathtt{c:} & \left.\begin{array}{l} \mathtt{3\underline{a}ecb}\ldots \\ \mathtt{3\underline{a}eeb}\ldots \\ \mathtt{4\underline{c}abd}\ldots \end{array}\right\} \to\to \mathtt{2\_abcc}\ldots, \left\{\begin{array}{l} \mathtt{2caa\underline{b}}\ldots \\ \mathtt{3caa\underline{b}}\ldots \end{array}\right\} \end{array}\right. \tag{247}$$

$$\mathtt{3\underline{a}e}\ldots \to\to \mathtt{2\_ab}\ldots \Rightarrow \left\{ \begin{array}{l} \mathtt{a:} \left.\begin{array}{l} \mathtt{5\underline{c}ae}\ldots \\ \mathtt{5\underline{e}ae}\ldots \end{array}\right\} \to\to \mathtt{3dbc}\ldots \\ \mathtt{b:4\underline{b}eb}\ldots \to\to \mathtt{4\_cab}\ldots \\ \mathtt{c:} \left.\begin{array}{l} \mathtt{4\underline{c}ae}\ldots \\ \mathtt{3\underline{a}eb}\ldots \end{array}\right\} \to\to \mathtt{3abc}\ldots \end{array}\right. , \{\mathtt{5}\alpha\mathtt{a\underline{b}}\ldots\} \tag{248}$$

$$\mathtt{3\underline{a}ed}\ldots \to\to \mathtt{2\_aba}\ldots \Rightarrow \left\{ \begin{array}{ll} \mathtt{a:} & \left.\begin{array}{l} \mathtt{5\underline{c}aed}\ldots \\ \mathtt{5\underline{e}aed}\ldots \end{array}\right\} \to\to \mathtt{4dbcc}\ldots \\ \mathtt{c:} & \left.\begin{array}{l} \mathtt{4\underline{c}aed}\ldots \\ \mathtt{3aebd}\ldots \end{array}\right\} \to\to \mathtt{4abcc}\ldots \end{array}\right. , \emptyset \tag{249}$$

$$\mathtt{3\underline{e}a}\ldots \to\to \mathtt{4\_ca}\ldots \Rightarrow \left\{ \begin{array}{l} \mathtt{b:3\underline{e}ea}\ldots \to\to \mathtt{4bcc}\ldots \\ \mathtt{c:4\underline{c}ea}\ldots \to\to \mathtt{1abc}\ldots \end{array}\right. , \{\mathtt{1}\alpha\mathtt{e\underline{c}}\ldots\} \tag{250}$$

$$\mathtt{3\underline{e}aa\ldots} \to\to \mathtt{4\_cab\ldots} \Rightarrow \left\{\begin{array}{l} \mathtt{b : 3\underline{e}eaa\ldots} \to\to \mathtt{4bccb\ldots} \\ \mathtt{c : 4\underline{c}eaa\ldots} \to\to \mathtt{2abdb\ldots} \end{array}\right., \emptyset \tag{251}$$

$$\mathtt{3\underline{e}e\ldots} \to\to \mathtt{4\_ce\ldots} \Rightarrow \left\{\begin{array}{l} \mathtt{b : 3\underline{e}ee\ldots} \to\to \mathtt{3bcb\ldots} \\ \mathtt{c : 4\underline{c}ee\ldots} \to\to \mathtt{3\_bcc\ldots} \end{array}\right., \{\mathtt{5}\alpha\mathtt{e\underline{b}\ldots}\} \tag{252}$$

$$\mathtt{3\underline{e}ed\ldots} \to\to \mathtt{4\_cec\ldots} \Rightarrow \mathtt{b : 3\underline{e}eed\ldots} \to\to \mathtt{1babc\ldots}, \emptyset \tag{253}$$

$$\mathtt{4\underline{b}b\ldots} \to\to \mathtt{4\_ce\ldots} \Rightarrow \left\{\begin{array}{l} \mathtt{b : 4\underline{b}bb\ldots} \to\to \mathtt{3bcb\ldots} \\ \mathtt{c : 3\underline{a}bb\ldots} \to\to \mathtt{3\_bcc\ldots} \end{array}\right., \emptyset \tag{254}$$

$$\mathtt{4\underline{b}bd\ldots} \to\to \mathtt{4\_cec\ldots} \Rightarrow \mathtt{b : 4\underline{b}bbd\ldots} \to\to \mathtt{1bcbc\ldots}, \emptyset \tag{255}$$

$$\mathtt{4\underline{b}e\ldots} \to\to \mathtt{4\_cb\ldots} \Rightarrow \left\{\begin{array}{l} \mathtt{b : 4\underline{b}be\ldots} \to\to \mathtt{2bab\ldots} \\ \mathtt{c : 3\underline{a}be\ldots} \to\to \mathtt{3abc\ldots} \end{array}\right., \emptyset \tag{256}$$

$$\mathtt{4\underline{b}ed\ldots} \to\to \mathtt{4\_cba\ldots} \Rightarrow \left\{\begin{array}{l} \mathtt{b : 4\underline{b}bed\ldots} \to\to \mathtt{1babc\ldots} \\ \mathtt{c : 3\underline{a}bed\ldots} \to\to \mathtt{4abcc\ldots} \end{array}\right., \emptyset \tag{257}$$

$$\mathtt{4\underline{c}\ldots} \to\to \mathtt{2\_a\ldots} \Rightarrow \left\{\begin{array}{l} \mathtt{b : 4\underline{b}c\ldots} \to\to \mathtt{4\_ca\ldots} \\ \mathtt{c : 3\underline{a}c\ldots} \to\to \mathtt{2ab\ldots} \end{array}\right. \tag{258}$$

$$\mathtt{4\underline{c}e\ldots} \to\to \mathtt{2\_ae\ldots} \Rightarrow \mathtt{c : 3\underline{a}ce\ldots} \to\to \mathtt{3\_bcc\ldots}, \emptyset \tag{259}$$

$$\mathtt{5}\underline{\beta}\mathtt{e\ldots} \to\to \mathtt{3\_bc\ldots} \Rightarrow \mathtt{a : 4\underline{e}}\beta\mathtt{e\ldots} \to\to \mathtt{3cbc\ldots}, \emptyset \text{ for } \beta \in \{\mathtt{c}, \mathtt{e}\} \tag{260}$$

$$\mathtt{5}\underline{\beta}\mathtt{ec\ldots} \to\to \mathtt{3\_bca\ldots} \Rightarrow \mathtt{a : 4\underline{e}}\beta\mathtt{ec\ldots} \to\to \mathtt{4cbcc\ldots}, \emptyset \text{ for } \beta \in \{\mathtt{c}, \mathtt{e}\} \tag{261}$$

$$\mathtt{1\ldots\underline{a}} \to\to \mathtt{1\ldots d\_} \Rightarrow \emptyset, \emptyset \tag{262}$$

$$\mathtt{1\ldots b\underline{a}} \to\to \mathtt{1\ldots ba\_} \Rightarrow \emptyset, \emptyset \tag{263}$$

$$\mathtt{1\ldots b\underline{a}} \to\to \mathtt{1\ldots bd\_} \Rightarrow \emptyset, \emptyset \tag{264}$$

$$\mathtt{1\ldots b\underline{a}} \to\to \mathtt{2\ldots db\_} \Rightarrow \emptyset, \emptyset \tag{265}$$

$$\mathtt{1\ldots e\underline{a}} \to\to \mathtt{2\ldots db\_} \Rightarrow \emptyset, \emptyset \tag{266}$$

$$\mathtt{1}\ldots\mathtt{a}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{db}_{\_} \Rightarrow \emptyset, \{\mathtt{2}\ldots\underline{\mathtt{d}}\mathtt{b}\alpha\} \tag{267}$$

$$\mathtt{1}\ldots\mathtt{ea}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{bdb}_{\_} \Rightarrow \emptyset, \emptyset \tag{268}$$

$$\mathtt{1}\ldots\mathtt{d}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{ba}_{\_} \Rightarrow \emptyset, \{\mathtt{2}\ldots\underline{\mathtt{c}}\mathtt{b}\alpha\} \tag{269}$$

$$\mathtt{1}\ldots\mathtt{cd}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{dba}_{\_} \Rightarrow \emptyset, \emptyset \tag{270}$$

$$\mathtt{1}\ldots\mathtt{b}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{bc}_{\_} \Rightarrow \emptyset, \emptyset \tag{271}$$

$$\mathtt{1}\ldots\mathtt{b}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{ac}_{\_} \Rightarrow \emptyset, \emptyset \tag{272}$$

$$\mathtt{1}\ldots\mathtt{b}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{cc}_{\_} \Rightarrow \emptyset, \emptyset \tag{273}$$

$$\mathtt{1}\ldots\mathtt{c}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{bc}_{\_} \Rightarrow \emptyset, \{\mathtt{2}\ldots\underline{\mathtt{a}}\mathtt{c}\alpha\} \tag{274}$$

$$\mathtt{1}\ldots\mathtt{ac}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{dbc}_{\_} \Rightarrow \emptyset, \emptyset \tag{275}$$

$$\mathtt{1}\ldots\mathtt{d}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{bc}_{\_} \Rightarrow \emptyset, \{\mathtt{2}\ldots\underline{\mathtt{c}}\mathtt{c}\alpha\} \tag{276}$$

$$\mathtt{1}\ldots\mathtt{ad}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{dbc}_{\_} \Rightarrow \emptyset, \emptyset \tag{277}$$

$$\mathtt{2}\ldots\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{d}_{\_} \Rightarrow \left\{ \begin{array}{l} \mathtt{a}:\mathtt{3}\ldots\mathtt{b}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab}_{\_} \\ \mathtt{d}:\mathtt{1}\ldots\mathtt{b}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{db}_{\_} \\ \mathtt{e}:\mathtt{5}\ldots\mathtt{b}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{db}_{\_} \end{array} \right. \tag{278}$$

$$\mathtt{2}\ldots\mathtt{a}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab}_{\_} \Rightarrow \mathtt{e}:\mathtt{5}\ldots\mathtt{ab}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{bcc}, \emptyset \tag{279}$$

$$\mathtt{2}\ldots\mathtt{ea}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{bab}_{\_} \Rightarrow \mathtt{e}:\mathtt{5}\ldots\mathtt{eab}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{cbcc}, \emptyset \tag{280}$$

$$\mathtt{2}\ldots\mathtt{a}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{bc}_{\_} \Rightarrow \emptyset, \emptyset \tag{281}$$

$$\mathtt{2}\ldots\mathtt{d}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{bd}_{\_} \Rightarrow \emptyset, \emptyset \tag{282}$$

$$\mathtt{2}\ldots\mathtt{b}\underline{\mathtt{e}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{bd}_{\_} \Rightarrow \emptyset, \left\{ \begin{array}{l} \mathtt{1}\ldots\underline{\mathtt{d}}\mathtt{e}\alpha \\ \mathtt{1}\ldots\underline{\mathtt{e}}\mathtt{e}\alpha \end{array} \right\} \tag{283}$$

$$2\ldots\mathtt{ab}\underline{\mathtt{e}} \to\to 1\ldots\mathtt{dbd}\_ \Rightarrow \mathtt{c}: \left.\begin{array}{l} 3\ldots\mathtt{cba}\underline{\mathtt{d}}\\ 4\ldots\mathtt{cba}\underline{\mathtt{d}}\\ 3\ldots\mathtt{ccb}\underline{\mathtt{d}}\\ 4\ldots\mathtt{ccb}\underline{\mathtt{d}}\\ 3\ldots\mathtt{cbb}\underline{\mathtt{d}}\\ 4\ldots\mathtt{cbb}\underline{\mathtt{d}} \end{array}\right\} \to\to 5\ldots\mathtt{ceca}, \left\{\begin{array}{l} 2\ldots\underline{\mathtt{d}}\mathtt{de}\alpha\\ 2\ldots\underline{\mathtt{c}}\mathtt{ae}\alpha\\ 4\ldots\mathtt{cce}\alpha\\ 4\ldots\mathtt{cbe}\alpha \end{array}\right\} \tag{284}$$

$$2\ldots\underline{\mathtt{e}} \to\to 2\ldots\mathtt{b}\_ \Rightarrow \left\{\begin{array}{l} \mathtt{a}: 3\ldots\mathtt{e}\underline{\mathtt{c}} \to\to 1\ldots\mathtt{bc}\_\\ \mathtt{d}: 1\ldots\mathtt{e}\underline{\mathtt{a}} \to\to 1\ldots\mathtt{ba}\_\\ \mathtt{e}: 5\ldots\mathtt{e}\underline{\mathtt{a}} \to\to 4\ldots\mathtt{cc} \end{array}\right. \tag{285}$$

$$2\ldots\mathtt{b}\underline{\mathtt{e}} \to\to 2\ldots\mathtt{ab}\_ \Rightarrow \mathtt{e}: 5\ldots\mathtt{be}\underline{\mathtt{a}} \to\to 3\ldots\mathtt{bcc}, \left\{\begin{array}{l} 1\ldots\underline{\mathtt{d}}\mathtt{e}\alpha\\ 1\ldots\underline{\mathtt{e}}\mathtt{e}\alpha \end{array}\right\} \tag{286}$$

$$2\ldots\mathtt{bb}\underline{\mathtt{e}} \to\to 2\ldots\mathtt{bab}\_ \Rightarrow \mathtt{e}: 5\ldots\mathtt{bbe}\underline{\mathtt{a}} \to\to 4\ldots\mathtt{cbcc}, \emptyset \tag{287}$$

$$2\ldots\mathtt{b}\underline{\mathtt{e}} \to\to 2\ldots\mathtt{db}\_ \Rightarrow \mathtt{e}: 5\ldots\mathtt{be}\underline{\mathtt{a}} \to\to 5\ldots\mathtt{ccc}, \left\{\begin{array}{l} 1\ldots\underline{\mathtt{d}}\mathtt{e}\alpha\\ 1\ldots\underline{\mathtt{e}}\mathtt{e}\alpha \end{array}\right\} \tag{288}$$

$$2\ldots\mathtt{eb}\underline{\mathtt{e}} \to\to 2\ldots\mathtt{adb}\_ \Rightarrow \mathtt{e}: 5\ldots\mathtt{ebe}\underline{\mathtt{a}} \to\to 2\ldots\mathtt{eccc}, \emptyset \tag{289}$$

$$2\ldots\underline{\mathtt{e}} \to\to 1\ldots\mathtt{d}\_ \Rightarrow \left\{\begin{array}{l} \mathtt{a}: 3\ldots\mathtt{e}\underline{\mathtt{c}} \to\to 2\ldots\mathtt{ab}\_\\ \mathtt{d}: 1\ldots\mathtt{e}\underline{\mathtt{a}} \to\to 2\ldots\mathtt{db}\_\\ \mathtt{e}: 5\ldots\mathtt{e}\underline{\mathtt{a}} \to\to 2\ldots\mathtt{db}\_ \end{array}\right. \tag{290}$$

$$2\ldots\gamma\underline{\mathtt{e}} \to\to 1\ldots\mathtt{bd}\_ \Rightarrow \emptyset, \emptyset \text{ for } \gamma \in \{\mathtt{c}, \mathtt{d}\} \tag{291}$$

$$2\ldots\mathtt{a}\gamma\underline{\mathtt{e}} \to\to 1\ldots\mathtt{dbd}\_ \Rightarrow \emptyset, \emptyset \text{ for } \gamma \in \{\mathtt{c}, \mathtt{d}\} \tag{292}$$

$$3\ldots\mathtt{a}\underline{\mathtt{b}} \to\to 2\ldots\mathtt{ab}\_ \Rightarrow \mathtt{e}: 5\ldots\mathtt{ab}\underline{\mathtt{b}} \to\to 3\ldots\mathtt{bcc}, \left\{\begin{array}{l} 5\ldots\underline{\mathtt{c}}\mathtt{b}\alpha\\ 5\ldots\underline{\mathtt{e}}\mathtt{b}\alpha \end{array}\right\} \tag{293}$$

$$3\ldots\mathtt{ea}\underline{\mathtt{b}} \to\to 2\ldots\mathtt{bab}\_ \Rightarrow \mathtt{e}: 5\ldots\mathtt{eab}\underline{\mathtt{b}} \to\to 4\ldots\mathtt{cbcc}, \emptyset \tag{294}$$

$$3\ldots\mathtt{a}\underline{\mathtt{b}} \to\to 3\ldots\mathtt{bc}\_ \Rightarrow \emptyset, \left\{\begin{array}{l} 5\ldots\underline{\mathtt{c}}\mathtt{b}\alpha\\ 5\ldots\underline{\mathtt{e}}\mathtt{b}\alpha \end{array}\right\} \tag{295}$$

$$\mathtt{3}\ldots\mathtt{\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{d}\_ \Rightarrow \left\{\begin{array}{l} \mathtt{a:1}\ldots\mathtt{b\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab}\_ \\ \mathtt{b:4}\ldots\mathtt{b\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{5}\ldots\mathtt{cd} \\ \mathtt{c:2}\ldots\mathtt{b\underline{e}} \rightarrow\!\!\!\rightarrow \mathtt{5}\ldots\mathtt{ca} \\ \mathtt{e:5}\ldots\mathtt{b\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{db}\_ \end{array}\right. \tag{296}$$

$$\mathtt{3}\ldots\mathtt{d\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bd}\_ \Rightarrow \left\{\begin{array}{l} \mathtt{b:4}\ldots\mathtt{db\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{ecd} \\ \mathtt{c:2}\ldots\mathtt{db\underline{e}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{eca} \end{array}\right., \emptyset \tag{297}$$

$$\mathtt{3}\ldots\mathtt{cd\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{dbd}\_ \Rightarrow \left\{\begin{array}{l} \mathtt{b:4}\ldots\mathtt{cdb\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{5}\ldots\mathtt{cecd} \\ \mathtt{c:2}\ldots\mathtt{cdb\underline{e}} \rightarrow\!\!\!\rightarrow \mathtt{5}\ldots\mathtt{ceca} \end{array}\right., \emptyset \tag{298}$$

$$\mathtt{3}\ldots\mathtt{\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{c}\_ \Rightarrow \left\{\begin{array}{l} \mathtt{a:1}\ldots\mathtt{c\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{db}\_ \\ \mathtt{b:4}\ldots\mathtt{c\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{db}\_ \\ \mathtt{c:2}\ldots\mathtt{c\underline{e}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{aa} \\ \mathtt{e:5}\ldots\mathtt{c\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{cb}\_ \end{array}\right. \tag{299}$$

$$\mathtt{3}\ldots\mathtt{b\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bc}\_ \Rightarrow \left\{\begin{array}{l} \mathtt{c:2}\ldots\mathtt{bc\underline{e}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{caa} \\ \mathtt{a:3}\ldots\mathtt{ee\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{bdb}\_ \\ \mathtt{d:1}\ldots\mathtt{ee\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{bcb}\_ \\ \mathtt{e:5}\ldots\mathtt{ee\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{bcb}\_ \end{array}\right., \{\mathtt{3}\ldots\mathtt{\underline{e}c}\alpha\} \tag{300}$$

$$\mathtt{3}\ldots\mathtt{cb\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{abc}\_ \Rightarrow \mathtt{c:2}\ldots\mathtt{cbc\underline{e}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bcaa}, \{\mathtt{4}\ldots\mathtt{\underline{c}ec}\alpha\} \tag{301}$$

$$\mathtt{3}\ldots\mathtt{\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{c}\_ \Rightarrow \left\{\begin{array}{l} \mathtt{a:1}\ldots\mathtt{c\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab} \\ \mathtt{b:4}\ldots\mathtt{c\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{cb}\_ \\ \mathtt{c:2}\ldots\mathtt{c\underline{e}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{cc}\_ \\ \mathtt{e:5}\ldots\mathtt{c\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{5}\ldots\mathtt{ca}\_ \end{array}\right. \tag{302}$$

$$\mathtt{3}\ldots\mathtt{b\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{ac}\_ \Rightarrow \left\{\begin{array}{l} \mathtt{a:} \left.\begin{array}{c} \mathtt{1}\ldots\mathtt{bc\underline{c}} \\ \mathtt{3}\ldots\mathtt{ee\underline{c}} \end{array}\right\} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{dbc}\_ \\ \mathtt{d:1}\ldots\mathtt{ee\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{adb}\_ \\ \mathtt{e:5}\ldots\mathtt{ee\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{5}\ldots\mathtt{aca}\_ \end{array}\right., \{\mathtt{3}\ldots\mathtt{\underline{e}c}\alpha\} \tag{303}$$

$$\mathtt{3}\ldots\mathtt{eb\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{aac}\_ \Rightarrow \emptyset, \emptyset \tag{304}$$

$$\mathtt{3}\ldots\mathtt{b\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{cc}\_ \Rightarrow \left\{\begin{array}{l} \mathtt{a:3}\ldots\mathtt{ee\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{abc}\_ \\ \mathtt{d:1}\ldots\mathtt{ee\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{cdb}\_ \\ \mathtt{e:5}\ldots\mathtt{ee\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{5}\ldots\mathtt{cca}\_ \\ \mathtt{a:1}\ldots\mathtt{bc\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{abc}\_ \end{array}\right., \{\mathtt{3}\ldots\mathtt{\underline{e}c}\alpha\} \tag{305}$$

$$\mathtt{3}\ldots\mathtt{e}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{cc}\_ \Rightarrow \mathtt{a}:\mathtt{1}\ldots\mathtt{ec}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{abc}\_, \emptyset \tag{306}$$

$$\mathtt{3}\ldots\mathtt{ae}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{ccc}\_ \Rightarrow \emptyset, \emptyset \tag{307}$$

$$\mathtt{3}\ldots\underline{\mathtt{d}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{d}\_ \Rightarrow \left\{ \begin{array}{l} \mathtt{a}:\mathtt{1}\ldots\mathtt{d}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab}\_ \\ \mathtt{b}:\mathtt{4}\ldots\mathtt{d}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{5}\ldots\mathtt{cd} \\ \mathtt{c}:\mathtt{2}\ldots\mathtt{d}\underline{\mathtt{e}} \rightarrow\!\!\rightarrow \mathtt{5}\ldots\mathtt{ca} \\ \mathtt{e}:\mathtt{5}\ldots\mathtt{d}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{db}\_ \end{array} \right. \tag{308}$$

$$\mathtt{3}\ldots\mathtt{a}\underline{\mathtt{d}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{bd}\_ \Rightarrow \left\{ \begin{array}{l} \mathtt{b}:\mathtt{4}\ldots\mathtt{ad}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{ecd} \\ \mathtt{c}:\mathtt{2}\ldots\mathtt{ad}\underline{\mathtt{e}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{eca} \end{array} \right., \left\{ \begin{array}{l} \mathtt{5}\ldots\underline{\mathtt{c}}\mathtt{d}\alpha \\ \mathtt{5}\ldots\underline{\mathtt{e}}\mathtt{d}\alpha \end{array} \right\} \tag{309}$$

$$\mathtt{3}\ldots\mathtt{aa}\underline{\mathtt{d}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{cbd}\_ \Rightarrow \left\{ \begin{array}{l} \mathtt{b}:\mathtt{4}\ldots\mathtt{aad}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{aecd} \\ \mathtt{c}:\mathtt{2}\ldots\mathtt{aad}\underline{\mathtt{e}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{aeca} \end{array} \right., \left\{ \begin{array}{l} \mathtt{4}\ldots\underline{\mathtt{e}}\mathtt{cd}\alpha \\ \mathtt{4}\ldots\underline{\mathtt{e}}\mathtt{ed}\alpha \end{array} \right\} \tag{310}$$

$$\mathtt{3}\ldots\mathtt{ca}\underline{\mathtt{d}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{dbd}\_ \Rightarrow \left\{ \begin{array}{l} \mathtt{b}:\mathtt{4}\ldots\mathtt{cad}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{5}\ldots\mathtt{cecd} \\ \mathtt{c}:\mathtt{2}\ldots\mathtt{cad}\underline{\mathtt{e}} \rightarrow\!\!\rightarrow \mathtt{5}\ldots\mathtt{ceca} \end{array} \right., \emptyset \tag{311}$$

$$\mathtt{3}\ldots\mathtt{da}\underline{\mathtt{d}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{abd}\_ \Rightarrow \left\{ \begin{array}{l} \mathtt{b}:\mathtt{4}\ldots\mathtt{dad}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{ccbd}\_ \\ \mathtt{c}:\mathtt{2}\ldots\mathtt{dad}\underline{\mathtt{e}} \rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{caac}\_ \end{array} \right., \emptyset \tag{312}$$

$$\mathtt{3}\ldots\mathtt{d}\underline{\mathtt{d}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{bd}\_ \Rightarrow \left\{ \begin{array}{l} \mathtt{b}:\mathtt{4}\ldots\mathtt{dd}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{ecd} \\ \mathtt{c}:\mathtt{2}\ldots\mathtt{dd}\underline{\mathtt{e}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{eca} \end{array} \right., \emptyset \tag{313}$$

$$\mathtt{3}\ldots\mathtt{ed}\underline{\mathtt{d}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{cbd}\_ \Rightarrow \left\{ \begin{array}{l} \mathtt{b}:\mathtt{4}\ldots\mathtt{edd}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{aecd} \\ \mathtt{c}:\mathtt{2}\ldots\mathtt{edd}\underline{\mathtt{e}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{aeca} \end{array} \right., \emptyset \tag{314}$$

$$\mathtt{3}\ldots\underline{\mathtt{d}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{a}\_ \Rightarrow \left\{ \begin{array}{l} \mathtt{a}:\mathtt{1}\ldots\mathtt{d}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{ac}\_ \\ \mathtt{b}:\mathtt{4}\ldots\mathtt{d}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{bc} \\ \mathtt{c}:\mathtt{2}\ldots\mathtt{d}\underline{\mathtt{e}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{db}\_ \\ \mathtt{e}:\mathtt{5}\ldots\mathtt{d}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{ab}\_ \end{array} \right. \tag{315}$$

$$\mathtt{3}\ldots\mathtt{b}\underline{\mathtt{d}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{aa}\_ \Rightarrow \mathtt{b}:\mathtt{4}\ldots\mathtt{bd}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{cbc}\_, \{\mathtt{3}\ldots\underline{\mathtt{e}}\mathtt{d}\alpha\} \tag{316}$$

$$\mathtt{3}\ldots\mathtt{ab}\underline{\mathtt{d}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{caa}\_ \Rightarrow \emptyset, \emptyset \tag{317}$$

$$\mathtt{3}\ldots\underline{\mathtt{d}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{c}\_ \Rightarrow \left\{ \begin{array}{l} \mathtt{a}:\mathtt{1}\ldots\mathtt{d}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{cc}\_ \\ \mathtt{b}:\mathtt{4}\ldots\mathtt{d}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab}\_ \\ \mathtt{c}:\mathtt{2}\ldots\mathtt{d}\underline{\mathtt{e}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab}\_ \\ \mathtt{e}:\mathtt{5}\ldots\mathtt{d}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{cb}\_ \end{array} \right. \tag{318}$$

$$\mathtt{3\ldots d\underline{d}} \to\!\!\!\to \mathtt{3\ldots cc\_} \Rightarrow \emptyset, \emptyset \tag{319}$$

$$\mathtt{3\ldots cd\underline{d}} \to\!\!\!\to \mathtt{3\ldots ccc\_} \Rightarrow \emptyset, \emptyset \tag{320}$$

$$\mathtt{3\ldots \underline{d}} \to\!\!\!\to \mathtt{4\ldots c\_} \Rightarrow \left\{ \begin{array}{l} \mathtt{a:1\ldots d\underline{c}} \to\!\!\!\to \mathtt{2\ldots ab} \\ \mathtt{b:4\ldots d\underline{a}} \to\!\!\!\to \mathtt{4\ldots cb\_} \\ \mathtt{c:2\ldots d\underline{e}} \to\!\!\!\to \mathtt{3\ldots cc\_} \\ \mathtt{e:5\ldots d\underline{b}} \to\!\!\!\to \mathtt{5\ldots ca\_} \end{array} \right. \tag{321}$$

$$\mathtt{3\ldots b\underline{d}} \to\!\!\!\to \mathtt{4\ldots cc\_} \Rightarrow \mathtt{a:1\ldots bd\underline{c}} \to\!\!\!\to \mathtt{3\ldots abc\_}, \{\mathtt{3\ldots \underline{e}d}\alpha\} \tag{322}$$

$$\mathtt{3\ldots eb\underline{d}} \to\!\!\!\to \mathtt{4\ldots bcc\_} \Rightarrow \emptyset, \emptyset \tag{323}$$

$$\mathtt{4\ldots \underline{a}} \to\!\!\!\to \mathtt{2\ldots b\_} \Rightarrow \left\{ \begin{array}{l} \mathtt{a:5\ldots a\underline{d}} \to\!\!\!\to \mathtt{1\ldots bc\_} \\ \mathtt{c:} \left. \begin{array}{c} \mathtt{2\ldots a\underline{b}} \\ \mathtt{3\ldots a\underline{b}} \end{array} \right\} \to\!\!\!\to \mathtt{1\ldots bd\_} \\ \mathtt{d:1\ldots a\underline{b}} \to\!\!\!\to \mathtt{1\ldots ba\_} \end{array} \right. \tag{324}$$

$$\mathtt{4\ldots b\underline{a}} \to\!\!\!\to \mathtt{2\ldots ab\_} \Rightarrow \mathtt{c:} \left. \begin{array}{c} \mathtt{3\ldots bd\underline{d}} \\ \mathtt{4\ldots bd\underline{d}} \end{array} \right\} \to\!\!\!\to \mathtt{1\ldots abd\_}, \{\mathtt{4\ldots \underline{b}a}\alpha\} \tag{325}$$

$$\mathtt{4\ldots ab\underline{a}} \to\!\!\!\to \mathtt{3\ldots dbc\_} \Rightarrow \emptyset, \emptyset \tag{326}$$

$$\mathtt{4\ldots bb\underline{a}} \to\!\!\!\to \mathtt{2\ldots bab\_} \Rightarrow \mathtt{a:} \left. \begin{array}{c} \mathtt{3\ldots bbd\underline{d}} \\ \mathtt{4\ldots bbd\underline{d}} \end{array} \right\} \to\!\!\!\to \mathtt{1\ldots babc\_}, \{\mathtt{4\ldots \underline{b}ba}\alpha\} \tag{327}$$

$$\mathtt{4\ldots b\underline{a}} \to\!\!\!\to \mathtt{3\ldots bc\_} \Rightarrow \mathtt{c:} \left. \begin{array}{c} \mathtt{3\ldots bd\underline{d}} \\ \mathtt{4\ldots bd\underline{d}} \end{array} \right\} \to\!\!\!\to \mathtt{2\ldots bab\_}, \{\mathtt{4\ldots \underline{b}a}\alpha\} \tag{328}$$

$$\mathtt{4\ldots eb\underline{a}} \to\!\!\!\to \mathtt{3\ldots cbc\_} \Rightarrow \emptyset, \emptyset \tag{329}$$

$$\mathtt{4\ldots c\underline{a}} \to\!\!\!\to \mathtt{3\ldots bc\_} \Rightarrow \emptyset, \left\{ \begin{array}{c} \mathtt{3\ldots \underline{a}a}\alpha \\ \mathtt{2\ldots \underline{d}c}\alpha \end{array} \right\} \tag{330}$$

$$\mathtt{4\ldots ac\underline{a}} \to\!\!\!\to \mathtt{3\ldots dbc\_} \Rightarrow \emptyset, \left\{ \begin{array}{c} \mathtt{5\ldots \underline{c}aa}\alpha \\ \mathtt{5\ldots \underline{e}aa}\alpha \end{array} \right\} \tag{331}$$

$$\mathtt{4\ldots d\underline{a}} \rightarrow\rightarrow \mathtt{3\ldots bc\_} \Rightarrow \emptyset, \emptyset \tag{332}$$

$$\mathtt{4\ldots \underline{d}} \rightarrow\rightarrow \mathtt{1\ldots d\_} \Rightarrow \left\{ \begin{array}{l} \mathtt{a}: \mathtt{5\ldots d\underline{d}} \rightarrow\rightarrow \mathtt{2\ldots ab\_} \\ \mathtt{c}: \left. \begin{array}{c} \mathtt{2\ldots d\underline{b}} \\ \mathtt{3\ldots d\underline{b}} \end{array} \right\} \rightarrow\rightarrow \mathtt{5\ldots ca} \\ \mathtt{d}: \mathtt{1\ldots d\underline{b}} \rightarrow\rightarrow \mathtt{2\ldots db\_} \end{array} \right. \tag{333}$$

$$\mathtt{4\ldots} \beta\mathtt{\underline{d}} \rightarrow\rightarrow \mathtt{1\ldots bd\_} \Rightarrow \mathtt{c}: \left. \begin{array}{c} \mathtt{2\ldots}\beta\mathtt{d\underline{b}} \\ \mathtt{3\ldots}\beta\mathtt{d\underline{b}} \end{array} \right\} \rightarrow\rightarrow \mathtt{3\ldots eca}, \emptyset \text{ for } \beta \in \{\mathtt{a}, \mathtt{d}\} \tag{334}$$

$$\mathtt{4\ldots aa\underline{d}} \rightarrow\rightarrow \mathtt{1\ldots cbd\_} \Rightarrow \mathtt{c}: \left. \begin{array}{c} \mathtt{2\ldots aad\underline{b}} \\ \mathtt{3\ldots aad\underline{b}} \end{array} \right\} \rightarrow\rightarrow \mathtt{2\ldots aeca}, \emptyset \tag{335}$$

$$\mathtt{4\ldots ca\underline{d}} \rightarrow\rightarrow \mathtt{1\ldots dbd\_} \Rightarrow \mathtt{c}: \left. \begin{array}{c} \mathtt{2\ldots cad\underline{b}} \\ \mathtt{3\ldots cad\underline{b}} \end{array} \right\} \rightarrow\rightarrow \mathtt{5\ldots ceca}, \emptyset \tag{336}$$

$$\mathtt{4\ldots da\underline{d}} \rightarrow\rightarrow \mathtt{1\ldots abd\_} \Rightarrow \mathtt{c}: \left. \begin{array}{c} \mathtt{2\ldots dad\underline{b}} \\ \mathtt{3\ldots dad\underline{b}} \end{array} \right\} \rightarrow\rightarrow \mathtt{4\ldots caac\_}, \emptyset \tag{337}$$

$$\mathtt{4\ldots ed\underline{d}} \rightarrow\rightarrow \mathtt{1\ldots cbd\_} \Rightarrow \mathtt{c}: \left. \begin{array}{c} \mathtt{2\ldots edd\underline{b}} \\ \mathtt{3\ldots edd\underline{b}} \end{array} \right\} \rightarrow\rightarrow \mathtt{2\ldots aeca}, \emptyset \tag{338}$$

$$\mathtt{4\ldots \underline{d}} \rightarrow\rightarrow \mathtt{3\ldots a\_} \Rightarrow \left\{ \begin{array}{l} \mathtt{a}: \mathtt{5\ldots d\underline{d}} \rightarrow\rightarrow \mathtt{4\ldots ac\_} \\ \mathtt{c}: \left. \begin{array}{c} \mathtt{2\ldots d\underline{b}} \\ \mathtt{3\ldots d\underline{b}} \end{array} \right\} \rightarrow\rightarrow \mathtt{2\ldots db\_} \\ \mathtt{d}: \mathtt{1\ldots d\underline{b}} \rightarrow\rightarrow \mathtt{2\ldots ec} \end{array} \right. \tag{339}$$

$$\mathtt{4\ldots b\underline{d}} \rightarrow\rightarrow \mathtt{3\ldots aa\_} \Rightarrow \mathtt{d}: \mathtt{1\ldots bd\underline{b}} \rightarrow\rightarrow \mathtt{1\ldots cbd\_}, \{\mathtt{4\ldots \underline{b}d}\alpha\} \tag{340}$$

$$\mathtt{4\ldots ab\underline{d}} \rightarrow\rightarrow \mathtt{3\ldots caa\_} \Rightarrow \emptyset, \emptyset \tag{341}$$

$$\mathtt{4\ldots \underline{d}} \rightarrow\rightarrow \mathtt{3\ldots c\_} \Rightarrow \left\{ \begin{array}{l} \mathtt{a}: \mathtt{5\ldots d\underline{d}} \rightarrow\rightarrow \mathtt{4\ldots cc\_} \\ \mathtt{c}: \left. \begin{array}{c} \mathtt{2\ldots d\underline{b}} \\ \mathtt{3\ldots d\underline{b}} \end{array} \right\} \rightarrow\rightarrow \mathtt{2\ldots ab\_} \\ \mathtt{d}: \mathtt{1\ldots d\underline{b}} \rightarrow\rightarrow \mathtt{2\ldots db\_} \end{array} \right. \tag{342}$$

$$\mathtt{4\ldots \underline{d}} \rightarrow\rightarrow \mathtt{4\ldots c\_} \Rightarrow \left\{ \begin{array}{l} \mathtt{a}: \mathtt{5\ldots d\underline{d}} \rightarrow\rightarrow \mathtt{2\ldots ab} \\ \mathtt{c}: \left. \begin{array}{c} \mathtt{2\ldots d\underline{b}} \\ \mathtt{3\ldots d\underline{b}} \end{array} \right\} \rightarrow\rightarrow \mathtt{3\ldots cc\_} \\ \mathtt{d}: \mathtt{1\ldots d\underline{b}} \rightarrow\rightarrow \mathtt{2\ldots db\_} \end{array} \right. \tag{343}$$

$$\mathtt{4\ldots b\underline{d}} \rightarrow\rightarrow \mathtt{4\ldots cc\_} \Rightarrow \mathtt{a}: \mathtt{5\ldots bd\underline{d}} \rightarrow\rightarrow \mathtt{3\ldots abc\_}, \{\mathtt{4\ldots \underline{b}d}\alpha\} \tag{344}$$

$$\mathtt{4\ldots eb\underline{d}} \twoheadrightarrow \mathtt{4\ldots bcc\_} \Rightarrow \emptyset, \emptyset \tag{345}$$

$$\mathtt{4\ldots d\underline{d}} \twoheadrightarrow \mathtt{3\ldots cc\_} \Rightarrow \emptyset, \emptyset \tag{346}$$

$$\mathtt{5\ldots \underline{a}} \twoheadrightarrow \mathtt{3\ldots b\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l}\mathtt{3\ldots a\underline{d}}\\ \mathtt{4\ldots a\underline{d}}\end{array}\right\} \twoheadrightarrow \mathtt{4\ldots ca} \tag{347}$$

$$\mathtt{5\ldots b\underline{a}} \twoheadrightarrow \mathtt{3\ldots ab\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l}\mathtt{3\ldots ba\underline{d}}\\ \mathtt{4\ldots ba\underline{d}}\end{array}\right\} \twoheadrightarrow \mathtt{3\_bca\ldots}, \emptyset \tag{348}$$

$$\mathtt{5\ldots eb\underline{a}} \twoheadrightarrow \mathtt{3\ldots aab\_} \Rightarrow \emptyset, \emptyset \tag{349}$$

$$\mathtt{5\ldots}\beta\mathtt{\underline{a}} \twoheadrightarrow \mathtt{3\ldots cb\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l}\mathtt{3\ldots}\beta\mathtt{a\underline{d}}\\ \mathtt{4\ldots}\beta\mathtt{a\underline{d}}\end{array}\right\} \twoheadrightarrow \mathtt{1\ldots abc\_}, \emptyset \text{ for } \beta \in \{\mathtt{b}, \mathtt{e}\} \tag{350}$$

$$\mathtt{5\ldots b}\beta\mathtt{\underline{a}} \twoheadrightarrow \mathtt{3\ldots bcb\_} \Rightarrow \emptyset, \emptyset \text{ for } \beta \in \{\mathtt{b}, \mathtt{e}\} \tag{351}$$

$$\mathtt{5\ldots \underline{b}} \twoheadrightarrow \mathtt{1\ldots d\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l}\mathtt{3\ldots b\underline{d}}\\ \mathtt{4\ldots b\underline{d}}\end{array}\right\} \twoheadrightarrow \mathtt{5\ldots ca} \tag{352}$$

$$\mathtt{5\ldots c\underline{b}} \twoheadrightarrow \mathtt{1\ldots bd\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l}\mathtt{3\ldots cb\underline{d}}\\ \mathtt{4\ldots cb\underline{d}}\end{array}\right\} \twoheadrightarrow \mathtt{3\ldots eca}, \emptyset \tag{353}$$

$$\mathtt{5\ldots dc\underline{b}} \twoheadrightarrow \mathtt{1\ldots abd\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l}\mathtt{3\ldots dcbd}\\ \mathtt{4\ldots dcbd}\end{array}\right\} \twoheadrightarrow \mathtt{4\ldots caac\_}, \emptyset \tag{354}$$

$$\mathtt{5\ldots \underline{b}} \twoheadrightarrow \mathtt{3\ldots b\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l}\mathtt{3\ldots b\underline{d}}\\ \mathtt{4\ldots b\underline{d}}\end{array}\right\} \twoheadrightarrow \mathtt{4\ldots ca} \tag{355}$$

$$\mathtt{5\ldots b\underline{b}} \twoheadrightarrow \mathtt{3\ldots ab\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l}\mathtt{3\ldots bb\underline{d}}\\ \mathtt{4\ldots bb\underline{d}}\end{array}\right\} \twoheadrightarrow \mathtt{3\ldots bca}, \emptyset \tag{356}$$

$$\mathtt{5\ldots eb\underline{b}} \twoheadrightarrow \mathtt{3\ldots aab\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l}\mathtt{3\ldots ebb\underline{d}}\\ \mathtt{4\ldots ebb\underline{d}}\end{array}\right\} \twoheadrightarrow \mathtt{4\ldots cbcc\_}, \emptyset \tag{357}$$

$$\mathtt{5\ldots b\underline{b}} \twoheadrightarrow \mathtt{3\ldots cb\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l}\mathtt{3\ldots bb\underline{d}}\\ \mathtt{4\ldots bb\underline{d}}\end{array}\right\} \twoheadrightarrow \mathtt{1\ldots abc\_}, \emptyset \tag{358}$$

$$\mathtt{5\ldots bb\underline{b}} \twoheadrightarrow \mathtt{3\ldots bcb\_} \Rightarrow \emptyset, \emptyset \tag{359}$$

$$\mathtt{5\ldots \underline{d}} \twoheadrightarrow \mathtt{1\ldots c\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l}\mathtt{3\ldots d\underline{d}}\\ \mathtt{4\ldots d\underline{d}}\end{array}\right\} \twoheadrightarrow \mathtt{2\ldots aa} \tag{360}$$

$$\mathtt{5\ldots d\underline{d}} \twoheadrightarrow \mathtt{1\ldots bc\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l}\mathtt{3\ldots dd\underline{d}}\\ \mathtt{4\ldots dd\underline{d}}\end{array}\right\} \twoheadrightarrow \mathtt{4\ldots caa}, \emptyset \tag{361}$$

$$\mathtt{5\ldots cd\underline{d}} \twoheadrightarrow \mathtt{1\ldots dbc\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l}\mathtt{3\ldots cdd\underline{d}}\\ \mathtt{4\ldots cdd\underline{d}}\end{array}\right\} \twoheadrightarrow \mathtt{5\ldots ccaa}, \emptyset \tag{362}$$

$$\mathtt{5\ldots \underline{d}} \twoheadrightarrow \mathtt{3\ldots c\_} \Rightarrow \mathtt{c}: \left.\begin{array}{l}\mathtt{3\ldots d\underline{d}}\\ \mathtt{4\ldots d\underline{d}}\end{array}\right\} \twoheadrightarrow \mathtt{2\ldots ab\_} \tag{363}$$

$$\mathtt{5\ldots a\underline{d}} \twoheadrightarrow \mathtt{3\ldots bc\_} \Rightarrow \emptyset, \{\mathtt{4\ldots \underline{e}d}\alpha\} \tag{364}$$

$$\mathtt{5\ldots aa\underline{d}} \twoheadrightarrow \mathtt{3\ldots abc\_} \Rightarrow \emptyset, \emptyset \tag{365}$$

$$\mathtt{5\ldots a\underline{d}} \twoheadrightarrow \mathtt{4\ldots cc\_} \Rightarrow \emptyset, \{\mathtt{4\ldots \underline{e}d}\alpha\} \tag{366}$$

$$\mathtt{5\ldots ea\underline{d}} \twoheadrightarrow \mathtt{4\ldots bcc\_} \Rightarrow \emptyset, \emptyset \tag{367}$$

8.4 Additional results for generating the IRR(5)

$$\mathtt{3\underline{a}\ldots} \twoheadrightarrow \mathtt{5\_c\ldots} \Rightarrow \left\{\begin{array}{l} \mathtt{a}: \left.\begin{array}{l}\mathtt{5\underline{c}a\ldots}\\ \mathtt{5\underline{e}a\ldots}\end{array}\right\} \twoheadrightarrow \mathtt{2\_ec\ldots}\\ \mathtt{b}: \mathtt{3\underline{e}a\ldots} \twoheadrightarrow \mathtt{3\_ec\ldots}\\ \mathtt{c}: \mathtt{4\underline{c}a\ldots} \twoheadrightarrow \mathtt{2db\ldots}\end{array}\right. \tag{368}$$

$$\mathtt{3\underline{a}c\ldots} \twoheadrightarrow \mathtt{5\_ce\ldots} \Rightarrow \mathtt{c}: \mathtt{4\underline{c}ac} \twoheadrightarrow \mathtt{5\_ccc\ldots}, \{\mathtt{2}\alpha\mathtt{a\underline{e}\ldots}\} \tag{369}$$

$$\mathtt{3\underline{a}cd\ldots} \twoheadrightarrow \mathtt{5\_cec\ldots} \Rightarrow \emptyset, \{\mathtt{1}\alpha\mathtt{ae\underline{a}\ldots}\} \tag{370}$$

$$\mathtt{3\underline{a}cde\ldots} \twoheadrightarrow \mathtt{5\_ceca\ldots} \Rightarrow \emptyset, \emptyset \tag{371}$$

$$\mathtt{4\underline{c}\ldots} \twoheadrightarrow \mathtt{5\_c\ldots} \Rightarrow \left\{\begin{array}{l} \mathtt{b}: \mathtt{4\underline{b}c\ldots} \twoheadrightarrow \mathtt{3\_ec\ldots}\\ \mathtt{c}: \mathtt{3\underline{a}c\ldots} \twoheadrightarrow \mathtt{2db\ldots}\end{array}\right. \tag{372}$$

$$\mathtt{4\underline{c}a\ldots} \twoheadrightarrow \mathtt{5\_ce\ldots} \Rightarrow \mathtt{c}: \mathtt{3\underline{a}ca\ldots} \twoheadrightarrow \mathtt{5\_ccc\ldots}, \{\mathtt{5}\alpha\mathtt{c\underline{d}\ldots}\} \tag{373}$$

$$\mathtt{4\underline{c}ad\ldots} \twoheadrightarrow \mathtt{5\_cec\ldots} \Rightarrow \emptyset, \emptyset \tag{374}$$

$$\mathtt{4\underline{b}e\ldots} \twoheadrightarrow \mathtt{4\_ca\ldots} \Rightarrow \left\{\begin{array}{l} \mathtt{b}: \mathtt{4\underline{b}be\ldots} \twoheadrightarrow \mathtt{4bcc\ldots}\\ \mathtt{c}: \mathtt{3\underline{a}be\ldots} \twoheadrightarrow \mathtt{1abc\ldots}\end{array}\right. , \emptyset \tag{375}$$

$$\mathtt{4\underline{b}ec}\ldots \to\to \mathtt{4\_cae}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{b:4\underline{b}eeb}\ldots \to\to \mathtt{5bcca}\ldots \\ \mathtt{c:3\underline{a}bec}\ldots \to\to \mathtt{2abcb}\ldots \end{array}\right., \emptyset \tag{376}$$

$$\mathtt{4\underline{b}ecb}\ldots \to\to \mathtt{4\_caec}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{b:4\underline{b}eebb}\ldots \to\to \mathtt{3bcaaa}\ldots \\ \mathtt{c:3\underline{a}becb}\ldots \to\to \mathtt{1abcbd}\ldots \end{array}\right., \emptyset \tag{377}$$

$$\mathtt{4\underline{b}ee}\ldots \to\to \mathtt{4\_cae}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{b:4\underline{b}bee}\ldots \to\to \mathtt{5bcca}\ldots \\ \mathtt{c:3\underline{a}bee}\ldots \to\to \mathtt{2abcb}\ldots \end{array}\right., \emptyset \tag{378}$$

$$\mathtt{4\underline{b}eeb}\ldots \to\to \mathtt{4\_caec}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{b:4\underline{b}beeb}\ldots \to\to \mathtt{3bccaa}\ldots \\ \mathtt{c:3\underline{a}beeb}\ldots \to\to \mathtt{1abcbd}\ldots \end{array}\right., \emptyset \tag{379}$$

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$$\mathtt{2\underline{a}d}\ldots \to\to \mathtt{2\_ae}\ldots \Rightarrow \left.\begin{array}{l} \mathtt{1\underline{d}da}\ldots \\ \mathtt{1\underline{e}da}\ldots \end{array}\right\} \to\to \mathtt{4\_cae}\ldots, \left\{\begin{array}{l} \mathtt{1}\alpha\mathtt{a\underline{a}}\ldots \\ \mathtt{3}\alpha\mathtt{d\underline{c}}\ldots \end{array}\right\} \tag{380}$$

$$\mathtt{2\underline{a}e}\ldots \to\to \mathtt{2\_ae}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{4\underline{b}ea}\ldots \to\to \mathtt{4\_cae}\ldots \\ \mathtt{3\underline{a}ea}\ldots \to\to \mathtt{3\_bcc}\ldots \end{array}\right., \left\{\begin{array}{l} \mathtt{5}\alpha\mathtt{a\underline{a}}\ldots \\ \mathtt{5}\alpha\mathtt{e\underline{d}}\ldots \end{array}\right\} \tag{381}$$

$$\mathtt{2\underline{c}d}\ldots \to\to \mathtt{5\_ce}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{a:} \left.\begin{array}{l} \mathtt{4\underline{e}cc}\ldots \\ \mathtt{4\underline{e}ec}\ldots \\ \mathtt{5\underline{c}ad}\ldots \\ \mathtt{5\underline{e}ad}\ldots \end{array}\right\} \to\to \mathtt{2\_ece}\ldots \\ \mathtt{b:} \left.\begin{array}{l} \mathtt{1\underline{d}ab}\ldots \\ \mathtt{1\underline{e}ab}\ldots \\ \mathtt{1\underline{d}aa}\ldots \\ \mathtt{1\underline{e}aa}\ldots \\ \mathtt{3\underline{e}ad}\ldots \\ \mathtt{4\underline{b}cd}\ldots \end{array}\right\} \to\to \mathtt{3\_ece}\ldots \\ \mathtt{c:} \left.\begin{array}{l} \mathtt{3\underline{a}cd}\ldots \\ \mathtt{4\underline{c}ad}\ldots \end{array}\right\} \to\to \mathtt{5\_ccc}\ldots \end{array}\right., \left\{\begin{array}{l} \mathtt{1}\alpha\mathtt{c\underline{a}}\ldots \\ \mathtt{3}\alpha\mathtt{a\underline{c}}\ldots \\ \mathtt{3}\alpha\mathtt{c\underline{d}}\ldots \\ \mathtt{4}\alpha\mathtt{c\underline{d}}\ldots \\ \mathtt{3}\alpha\mathtt{e\underline{d}}\ldots \\ \mathtt{4}\alpha\mathtt{e\underline{d}}\ldots \\ \mathtt{1}\alpha\mathtt{c\underline{b}}\ldots \end{array}\right\} \tag{382}$$

$$\mathtt{2\underline{c}e}\ldots \to\to \mathtt{5\_ce}\ldots \Rightarrow \emptyset, \{\mathtt{5}\alpha\mathtt{c\underline{a}}\ldots\} \tag{383}$$

$$\mathtt{4\underline{e}}\ldots \to\to \mathtt{2\_e}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{b:4\underline{b}e}\ldots \to\to \mathtt{4\_ce}\ldots \\ \mathtt{c:3\underline{a}e}\ldots \to\to \mathtt{2db}\ldots \end{array}\right., \emptyset \tag{384}$$

$$\mathtt{4}\underline{\mathtt{e}}\mathtt{c}\ldots \to\to \mathtt{2\_ec}\ldots \Rightarrow \mathtt{c} : \mathtt{3}\underline{\mathtt{a}}\mathtt{ec}\ldots \to\to \mathtt{1dbd}\ldots, \{\mathtt{2}\alpha\mathtt{e}\underline{\mathtt{b}}\ldots, \mathtt{3}\alpha\mathtt{e}\underline{\mathtt{b}}\ldots\} \tag{385}$$

$$\mathtt{4}\underline{\mathtt{e}}\mathtt{cc}\ldots \to\to \mathtt{2\_ecc}\ldots \Rightarrow \mathtt{c} : \mathtt{3}\underline{\mathtt{a}}\mathtt{ecc}\ldots \to\to \mathtt{5\_ceca}\ldots, \{\mathtt{2}\alpha\mathtt{eb}\underline{\mathtt{e}}\ldots\} \tag{386}$$

$$\mathtt{4}\underline{\mathtt{e}}\mathtt{e}\ldots \to\to \mathtt{2\_ec}\ldots \Rightarrow \mathtt{c} : \mathtt{3}\underline{\mathtt{a}}\mathtt{ee}\ldots \to\to \mathtt{1dbd}\ldots, \emptyset \tag{387}$$

$$\mathtt{4}\underline{\mathtt{e}}\mathtt{ec}\ldots \to\to \mathtt{2\_ecc}\ldots \Rightarrow \mathtt{c} : \mathtt{3}\underline{\mathtt{a}}\mathtt{eec}\ldots \to\to \mathtt{5\_ceca}\ldots, \emptyset \tag{388}$$

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$$\mathtt{3}\underline{\mathtt{e}}\mathtt{e}\ldots \to\to \mathtt{4\_ce}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{4}\underline{\mathtt{c}}\mathtt{ee}\ldots \to\to \mathtt{3\_bcc}\ldots \\ \mathtt{3}\underline{\mathtt{e}}\mathtt{ee}\ldots \to\to \mathtt{3bcb}\ldots \end{array}\right. \tag{389}$$

$$\mathtt{4}\underline{\mathtt{b}}\mathtt{b}\ldots \to\to \mathtt{4\_ce}\ldots \Rightarrow \mathtt{3}\underline{\mathtt{a}}\mathtt{bb}\ldots \to\to \mathtt{3\_bcc}\ldots \tag{390}$$

$$\mathtt{3}\underline{\mathtt{a}}\mathtt{b}\ldots \to\to \mathtt{2\_ae}\ldots \Rightarrow \left\{\begin{array}{l} \left.\begin{array}{l} \mathtt{5}\underline{\mathtt{c}}\mathtt{ab}\ldots \\ \mathtt{5}\underline{\mathtt{e}}\mathtt{ab}\ldots \end{array}\right\} \to\to \mathtt{5\_ccc}\ldots \\ \mathtt{4}\underline{\mathtt{c}}\mathtt{ab}\ldots \to\to \mathtt{3\_bcc}\ldots \end{array}\right. \tag{391}$$

$$\mathtt{3}\underline{\mathtt{a}}\ldots \to\to \mathtt{3\_b}\ldots \Rightarrow \left\{\begin{array}{l} \left.\begin{array}{l} \mathtt{5}\underline{\mathtt{c}}\mathtt{a}\ldots \\ \mathtt{5}\underline{\mathtt{e}}\mathtt{a}\ldots \end{array}\right\} \to\to \mathtt{4cb}\ldots \\ \mathtt{3}\underline{\mathtt{e}}\mathtt{a}\ldots \to\to \mathtt{4\_cb}\ldots \\ \mathtt{4}\underline{\mathtt{c}}\mathtt{a}\ldots \to\to \mathtt{2\_ab}\ldots \end{array}\right. \tag{392}$$

$$\mathtt{3}\underline{\mathtt{a}}\mathtt{e}\ldots \to\to \mathtt{3\_bc}\ldots \Rightarrow \left\{\begin{array}{l} \left.\begin{array}{l} \mathtt{5}\underline{\mathtt{c}}\mathtt{ae}\ldots \\ \mathtt{5}\underline{\mathtt{e}}\mathtt{ae}\ldots \end{array}\right\} \to\to \mathtt{3cbc}\ldots \\ \mathtt{4}\underline{\mathtt{b}}\mathtt{eb}\ldots \to\to \mathtt{4\_cbc}\ldots \\ \mathtt{3}\underline{\mathtt{a}}\mathtt{eb}\ldots \to\to \mathtt{2\_abc}\ldots \end{array}\right. , \{\mathtt{5}\alpha\mathtt{a}\underline{\mathtt{b}}\ldots\} \tag{393}$$

$$\mathtt{1}\underline{\gamma}\delta\ldots \to\to \text{ anything } \ldots \Rightarrow \emptyset, \emptyset \text{ for } \gamma \in \{\mathtt{d}, \mathtt{e}\}, \delta \in \{\mathtt{a}, \mathtt{c}, \mathtt{d}\} \tag{394}$$

$$\mathtt{1}\underline{\mathtt{d}}\ldots \to\to \mathtt{3\_e}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{2d}\underline{\mathtt{d}}\ldots \to\to \mathtt{5ca}\ldots \\ \mathtt{2}\underline{\mathtt{a}}\mathtt{d}\ldots \to\to \mathtt{2\_ae}\ldots \\ \mathtt{2}\underline{\mathtt{c}}\mathtt{d}\ldots \to\to \mathtt{5\_ce}\ldots \end{array}\right. \tag{395}$$

$$\mathtt{1}\underline{\mathtt{e}}\ldots \to\to \mathtt{3\_e}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{2}\underline{\mathtt{d}}\mathtt{e}\ldots \to\to \mathtt{5ca}\ldots \\ \mathtt{2}\underline{\mathtt{a}}\mathtt{e}\ldots \to\to \mathtt{2\_ae}\ldots \\ \mathtt{2}\underline{\mathtt{c}}\mathtt{e}\ldots \to\to \mathtt{5\_ce}\ldots \end{array}\right. \tag{396}$$

$$\mathtt{3\underline{a}e}\ldots \to\to \mathtt{2\_ab}\ldots \Rightarrow \mathtt{4\underline{b}eb}\ldots \to\to \mathtt{4\_cab}\ldots \tag{397}$$

$$\mathtt{4\underline{b}c}\ldots \to\to \mathtt{4\_ca}\ldots \Rightarrow \left\{ \begin{array}{l} \mathtt{4\underline{b}bc}\ldots \to\to \mathtt{4bcc}\ldots \\ \mathtt{3\underline{a}bc}\ldots \to\to \mathtt{1abc}\ldots \\ \mathtt{a}: \left. \begin{array}{l} \mathtt{2\underline{d}db}\ldots \\ \mathtt{2\underline{d}eb}\ldots \\ \mathtt{5\underline{c}eb}\ldots \\ \mathtt{5\underline{e}eb}\ldots \end{array} \right\} \to\to \mathtt{3\_bca}\ldots \\ \mathtt{b}: \mathtt{3\underline{e}eb}\ldots \to\to \mathtt{4bcc}\ldots \\ \mathtt{c}: \left. \begin{array}{l} \mathtt{2\underline{a}db}\ldots \\ \mathtt{2\underline{a}eb}\ldots \\ \mathtt{4\underline{c}eb}\ldots \end{array} \right\} \to\to \mathtt{1abc}\ldots \\ \mathtt{d}: \left. \begin{array}{l} \mathtt{2\underline{c}db}\ldots \\ \mathtt{2\underline{c}eb}\ldots \end{array} \right\} \to\to \mathtt{5\_cca}\ldots \end{array} \right. , \left\{ \begin{array}{l} \mathtt{2}\alpha\mathtt{b\underline{b}}\ldots \\ \mathtt{3}\alpha\mathtt{b\underline{b}}\ldots \\ \mathtt{4}\alpha\mathtt{e\underline{a}}\ldots \end{array} \right\} \tag{398}$$

$$\mathtt{4\underline{b}e}\ldots \to\to \mathtt{4\_ca}\ldots \Rightarrow \left\{ \begin{array}{l} \mathtt{4\underline{b}be}\ldots \to\to \mathtt{4bcc}\ldots \\ \mathtt{3\underline{a}be}\ldots \to\to \mathtt{1abc}\ldots \end{array} \right. , \emptyset \tag{399}$$

$$\mathtt{1}\ldots\mathtt{\underline{a}} \to\to \mathtt{1}\ldots\mathtt{a\_} \Rightarrow \emptyset \tag{400}$$

$$\mathtt{1}\ldots\mathtt{e\underline{a}} \to\to \mathtt{1}\ldots\mathtt{ba\_} \Rightarrow \emptyset, \emptyset \tag{401}$$

$$\mathtt{1}\ldots\mathtt{a\underline{b}} \to\to \mathtt{1}\ldots\mathtt{ba\_} \Rightarrow \emptyset, \{\mathtt{2}\ldots\mathtt{\underline{d}b}\alpha\} \tag{402}$$

$$\mathtt{1}\ldots\mathtt{d\underline{b}} \to\to \mathtt{2}\ldots\mathtt{db\_} \Rightarrow \emptyset, \{\mathtt{2}\ldots\mathtt{\underline{c}b}\alpha\} \tag{403}$$

$$\mathtt{1}\ldots\mathtt{b\underline{c}} \to\to \mathtt{2}\ldots\mathtt{ab\_} \Rightarrow \emptyset, \emptyset \tag{404}$$

$$\mathtt{1}\ldots\mathtt{c\underline{c}} \to\to \mathtt{2}\ldots\mathtt{db\_} \Rightarrow \emptyset, \{\mathtt{2}\ldots\mathtt{\underline{a}c}\alpha\} \tag{405}$$

$$\mathtt{1}\ldots\mathtt{d\underline{c}} \to\to \mathtt{2}\ldots\mathtt{ab\_} \Rightarrow \emptyset, \{\mathtt{2}\ldots\mathtt{\underline{c}c}\alpha\} \tag{406}$$

$$\mathtt{1}\ldots\mathtt{d\underline{c}} \to\to \mathtt{4}\ldots\mathtt{ac\_} \Rightarrow \emptyset, \{\mathtt{2}\ldots\mathtt{\underline{c}c}\alpha\} \tag{407}$$

$$\mathtt{1}\ldots\mathtt{d\underline{c}} \to\to \mathtt{4}\ldots\mathtt{cc\_} \Rightarrow \emptyset, \{\mathtt{2}\ldots\mathtt{\underline{c}c}\alpha\} \tag{408}$$

$$\mathtt{2}\ldots\mathtt{a\underline{b}} \to\to \mathtt{1}\ldots\mathtt{bd\_} \Rightarrow \emptyset, \emptyset \tag{409}$$

$$\mathtt{2}\ldots\mathtt{d\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab\_} \Rightarrow \mathtt{5}\ldots\mathtt{db\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bcc}, \emptyset \tag{410}$$

$$\mathtt{2}\ldots\mathtt{d\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{db\_} \Rightarrow \mathtt{5}\ldots\mathtt{db\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{5}\ldots\mathtt{ccc}, \emptyset \tag{411}$$

$$\mathtt{2}\ldots\mathtt{c\underline{e}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{cc\_} \Rightarrow \emptyset, \emptyset \tag{412}$$

$$\mathtt{2}\ldots\mathtt{d\underline{e}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab\_} \Rightarrow \mathtt{5}\ldots\mathtt{de\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bcc}, \emptyset \tag{413}$$

$$\mathtt{2}\ldots\mathtt{d\underline{e}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{db\_} \Rightarrow \mathtt{5}\ldots\mathtt{de\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{5}\ldots\mathtt{ccc}, \emptyset \tag{414}$$

$$\mathtt{3}\ldots\mathtt{a\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bd\_} \Rightarrow \left\{\begin{array}{l} \mathtt{4}\ldots\mathtt{ab\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{ecd} \\ \mathtt{2}\ldots\mathtt{ab\underline{e}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{eca} \end{array}\right., \left\{\begin{array}{l} \mathtt{5}\ldots\mathtt{\underline{c}b}\alpha \\ \mathtt{5}\ldots\mathtt{\underline{e}b}\alpha \end{array}\right\} \tag{415}$$

$$\mathtt{3}\ldots\mathtt{d\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab\_} \Rightarrow \mathtt{5}\ldots\mathtt{db\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bcc}, \emptyset \tag{416}$$

$$\mathtt{3}\ldots\mathtt{d\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{db\_} \Rightarrow \mathtt{5}\ldots\mathtt{db\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{5}\ldots\mathtt{ccc}, \emptyset \tag{417}$$

$$\mathtt{3}\ldots\mathtt{b\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab\_} \Rightarrow \left\{\begin{array}{l} \mathtt{5}\ldots\mathtt{bc\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bcc} \\ \mathtt{3}\ldots\mathtt{ee\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{abc\_} \\ \mathtt{1}\ldots\mathtt{ee\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{aba\_} \\ \mathtt{5}\ldots\mathtt{ee\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bcc} \end{array}\right., \{\mathtt{3}\ldots\mathtt{\underline{e}c}\alpha\} \tag{418}$$

$$\mathtt{3}\ldots\mathtt{e\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bc\_} \Rightarrow \mathtt{2}\ldots\mathtt{ec\underline{e}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{caa}, \emptyset \tag{419}$$

$$\mathtt{3}\ldots\mathtt{e\underline{c}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab\_} \Rightarrow \mathtt{5}\ldots\mathtt{ec\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bcc}, \emptyset \tag{420}$$

$$\mathtt{4}\ldots\mathtt{c\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{db\_} \Rightarrow \emptyset, \left\{\ \mathtt{3}\ldots\mathtt{\underline{a}a}\alpha, \mathtt{2}\ldots\mathtt{\underline{d}c}\alpha\ \right\} \tag{421}$$

$$\mathtt{4}\ldots\mathtt{c\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{cb\_} \Rightarrow \left\{\begin{array}{l} \mathtt{5}\ldots\mathtt{ca\underline{d}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{abc\_} \\ \mathtt{1}\ldots\mathtt{ca\underline{b}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{aec} \end{array}\right., \left\{\begin{array}{l} \mathtt{3}\ldots\mathtt{\underline{a}a}\alpha \\ \mathtt{2}\ldots\mathtt{\underline{d}c}\alpha \end{array}\right\} \tag{422}$$

$$\mathtt{4}\ldots\mathtt{d\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab\_} \Rightarrow \emptyset, \emptyset \tag{423}$$

$$\mathtt{5}\ldots\mathtt{b\underline{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{db\_} \Rightarrow \emptyset, \emptyset \tag{424}$$

$$\mathtt{5\ldots e\underline{a}} \rightarrow\rightarrow \mathtt{2\ldots db\_} \Rightarrow \emptyset, \emptyset \tag{425}$$

$$\mathtt{5\ldots \underline{b}} \rightarrow\rightarrow \mathtt{2\ldots b\_} \Rightarrow \left.\begin{array}{l}\mathtt{3\ldots b\underline{d}}\\ \mathtt{4\ldots b\underline{d}}\end{array}\right\} \rightarrow\rightarrow \mathtt{1\ldots bd\_} \tag{426}$$

$$\mathtt{5\ldots b\underline{b}} \rightarrow\rightarrow \mathtt{2\ldots db\_} \Rightarrow \emptyset, \emptyset \tag{427}$$

$$\mathtt{5\ldots c\underline{b}} \rightarrow\rightarrow \mathtt{2\ldots cb\_} \Rightarrow \emptyset, \emptyset \tag{428}$$

$$\mathtt{5\ldots c\underline{b}} \rightarrow\rightarrow \mathtt{5\ldots ca\_} \Rightarrow \emptyset, \emptyset \tag{429}$$

$$\mathtt{5\ldots d\underline{b}} \rightarrow\rightarrow \mathtt{2\ldots db\_} \Rightarrow \emptyset, \emptyset \tag{430}$$

$$\mathtt{5\ldots d\underline{b}} \rightarrow\rightarrow \mathtt{3\ldots ab\_} \Rightarrow \left.\begin{array}{l}\mathtt{3\ldots db\underline{d}}\\ \mathtt{4\ldots db\underline{d}}\end{array}\right\} \rightarrow\rightarrow \mathtt{3\ldots bca}, \emptyset \tag{431}$$

$$\mathtt{5\ldots d\underline{b}} \rightarrow\rightarrow \mathtt{3\ldots cb\_} \Rightarrow \left.\begin{array}{l}\mathtt{3\ldots db\underline{d}}\\ \mathtt{4\ldots db\underline{d}}\end{array}\right\} \rightarrow\rightarrow \mathtt{1\ldots abc\_}, \emptyset \tag{432}$$

$$\mathtt{5\ldots a\underline{d}} \rightarrow\rightarrow \mathtt{1\ldots bc\_} \Rightarrow \left.\begin{array}{l}\mathtt{3\ldots ad\underline{d}}\\ \mathtt{4\ldots ad\underline{d}}\end{array}\right\} \rightarrow\rightarrow \mathtt{4\ldots caa}, \{\mathtt{4\ldots \underline{e}d}\alpha\} \tag{433}$$

$$\mathtt{5\ldots d\underline{d}} \rightarrow\rightarrow \mathtt{2\ldots ab\_} \Rightarrow \emptyset, \emptyset \tag{434}$$

$$\mathtt{5\ldots d\underline{d}} \rightarrow\rightarrow \mathtt{4\ldots ac\_} \Rightarrow \emptyset, \emptyset \tag{435}$$

$$\mathtt{5\ldots d\underline{d}} \rightarrow\rightarrow \mathtt{4\ldots cc\_} \Rightarrow \emptyset, \emptyset \tag{436}$$

8.4.1 The main list

In this list, the IGR's are obtained by applying the context(s) to the original IGR's referred to if and only if the original IGR generates extendable IRR's, which happens if and only if the pointer is present in the notation as an underscore in its RHS. Thus for example, (540) below is only $\mathtt{2} \times \mathtt{3}$ IGR's instead of $\mathtt{3} \times \mathtt{3}$, the IGR having the RHS $\ldots\mathtt{2ab}$ playing no role.

$$\text{(131) context } (\mathtt{ac}, \mathtt{aa}) \tag{437}$$

$$\text{(164) context } (\mathtt{ac}, \mathtt{aa}) \tag{438}$$

$$\texttt{1}\underline{\texttt{d}}\texttt{ac}\ldots \to\to \texttt{4\_caa}\ldots \Rightarrow \texttt{2}\underline{\texttt{a}}\texttt{dac}\ldots \to\to \texttt{2abdb}\ldots \tag{439}$$

$$\texttt{1}\underline{\texttt{e}}\texttt{ac}\ldots \to\to \texttt{4\_caa}\ldots \Rightarrow \texttt{2}\underline{\texttt{a}}\texttt{eac}\ldots \to\to \texttt{2abdb}\ldots \tag{440}$$

$$(163) \text{ context } (\texttt{dc}, \texttt{ab}) \tag{441}$$

$$\texttt{1}\underline{\texttt{d}}\texttt{dc}\ldots \to\to \texttt{4\_cab}\ldots \Rightarrow \texttt{2}\underline{\texttt{a}}\texttt{ddc}\ldots \to\to \texttt{2abdb}\ldots \tag{442}$$

$$(164) \text{ context } (\texttt{dc}, \texttt{ab}) \tag{443}$$

$$\texttt{1}\underline{\texttt{e}}\texttt{dc}\ldots \to\to \texttt{4\_cab}\ldots \Rightarrow \texttt{2}\underline{\texttt{a}}\texttt{edc}\ldots \to\to \texttt{2abdb}\ldots \tag{444}$$

$$(163) \text{ context } (\texttt{db}, \texttt{bd}) \tag{445}$$

$$\texttt{1}\underline{\texttt{d}}\texttt{db}\ldots \to\to \texttt{4\_cbd}\ldots \Rightarrow \texttt{2}\underline{\texttt{a}}\texttt{ddb}\ldots \to\to \texttt{2abdb}\ldots \tag{446}$$

$$(164) \text{ context } (\texttt{db}, \texttt{bd}) \tag{447}$$

$$\texttt{1}\underline{\texttt{e}}\texttt{db}\ldots \to\to \texttt{4\_cbd}\ldots \Rightarrow \texttt{2}\underline{\texttt{a}}\texttt{edb}\ldots \to\to \texttt{2abdb}\ldots \tag{448}$$

$$(230) \text{ context } (\texttt{cb}, \texttt{cd}) \text{ for } \gamma = \texttt{d} \tag{449}$$

$$(230) \text{ context } (\texttt{cb}, \texttt{cd}) \text{ for } \gamma = \texttt{e} \tag{450}$$

$$(230) \text{ context } (\texttt{cc}, \texttt{ca}) \text{ for } \gamma = \texttt{d} \tag{451}$$

$$(230) \text{ context } (\texttt{cc}, \texttt{ca}) \text{ for } \gamma = \texttt{e} \tag{452}$$

$$\texttt{5}\underline{\texttt{c}}\texttt{aa}\ldots \to\to \texttt{3dbc}\ldots \Rightarrow \texttt{4}\underline{\texttt{e}}\texttt{caa}\ldots \to\to \texttt{3adbc}\ldots \tag{453}$$

$$(179) \text{ context } (\texttt{aa}, \texttt{ab}) \tag{454}$$

$$\texttt{3}\underline{\texttt{e}}\texttt{aa}\ldots \to\to \texttt{4\_cab}\ldots \Rightarrow \texttt{3}\underline{\texttt{e}}\texttt{eaa}\ldots \to\to \texttt{1babc}\ldots \tag{455}$$

$$\texttt{3}\underline{\texttt{e}}\texttt{aa}\ldots \to\to \texttt{4\_cab}\ldots \Rightarrow \texttt{4}\underline{\texttt{c}}\texttt{eaa}\ldots \to\to \texttt{2abdb}\ldots \tag{456}$$

$$\text{(179) context } (\texttt{ed}, \texttt{ec}) \tag{457}$$

$$\texttt{3}\underline{\texttt{e}}\texttt{ed}\ldots \to\to \texttt{4\_cec}\ldots \Rightarrow \texttt{3}\underline{\texttt{e}}\texttt{eed}\ldots \to\to \texttt{1babc}\ldots \tag{458}$$

$$\text{(258) context } (\texttt{ed}, \texttt{ec}) \tag{459}$$

$$\text{(390) context } (\texttt{d}, \texttt{c}) \tag{460}$$

$$\texttt{4}\underline{\texttt{b}}\texttt{bd}\ldots \to\to \texttt{4\_cec}\ldots \Rightarrow \texttt{4}\underline{\texttt{b}}\texttt{bbd}\ldots \to\to \texttt{1babc}\ldots \tag{461}$$

$$\text{(175) context } (\texttt{bd}, \texttt{ec}) \tag{462}$$

$$\text{(391) context } (\texttt{d}, \texttt{c}) \tag{463}$$

$$\text{(243) context } (\texttt{e}, \texttt{c}) \tag{464}$$

$$\text{(389) context } (\texttt{d}, \texttt{c}) \tag{465}$$

$$\text{(539) context } (\texttt{e}, \texttt{c}) \tag{466}$$

$$\text{(168) context } (\texttt{de}, \texttt{cc}) \tag{467}$$

$$\text{(168) context } (\texttt{ee}, \texttt{cc}) \tag{468}$$

$$\text{(173) context } \begin{matrix} (\texttt{de}, \texttt{cc}) \\ (\texttt{ee}, \texttt{cc}) \end{matrix} \tag{469}$$

$$\texttt{5}\underline{\texttt{c}}\texttt{ec}\ldots \to\to \texttt{3\_bca}\ldots \Rightarrow \texttt{4}\underline{\texttt{e}}\texttt{cec}\ldots \to\to \texttt{4cbcc}\ldots \tag{470}$$

$$\texttt{5}\underline{\texttt{e}}\texttt{ec}\ldots \to\to \texttt{3\_bca}\ldots \Rightarrow \texttt{4}\underline{\texttt{e}}\texttt{eec}\ldots \to\to \texttt{4cbcc}\ldots \tag{471}$$

$$\texttt{4}\underline{\texttt{b}}\texttt{ed}\ldots \to\to \texttt{4\_cba}\ldots \Rightarrow \begin{cases} \texttt{4}\underline{\texttt{b}}\texttt{bed}\ldots \to\to \texttt{1babc}\ldots \\ \texttt{3}\underline{\texttt{a}}\texttt{bed}\ldots \to\to \texttt{4abcc}\ldots \end{cases} \tag{472}$$

$$\text{(175) context } (\texttt{ed}, \texttt{ba}) \tag{473}$$

$$\text{(397) context } (\texttt{d}, \texttt{a}) \tag{474}$$

$$\mathtt{3}\underline{\mathtt{a}}\mathtt{ed}\ldots \rightarrow\!\!\rightarrow \mathtt{2\_aba}\ldots \Rightarrow \left\{\begin{array}{l} \left.\begin{array}{l}\mathtt{5}\underline{\mathtt{c}}\mathtt{aed}\ldots\\ \mathtt{5}\underline{\mathtt{e}}\mathtt{aed}\ldots\end{array}\right\}\rightarrow\!\!\rightarrow \mathtt{4dbcc}\ldots \\ \mathtt{4}\underline{\mathtt{c}}\mathtt{aed}\ldots \rightarrow\!\!\rightarrow \mathtt{4abcc}\ldots\\ \mathtt{3}\underline{\mathtt{a}}\mathtt{ebd}\ldots\rightarrow\!\!\rightarrow\mathtt{4abcc}\ldots\end{array}\right. \tag{475}$$

$$\mathtt{3}\ldots\mathtt{bb}\underline{\mathtt{c}} \rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{bcc\_} \Rightarrow \left\{\begin{array}{l} \left.\begin{array}{l}\mathtt{1}\ldots\mathtt{ebd}\underline{\mathtt{c}}\\ \mathtt{5}\ldots\mathtt{ebd}\underline{\mathtt{d}}\end{array}\right\}\rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{bccc\_} \\ \mathtt{4}\ldots\mathtt{ebd}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{bccb\_}\\ \left.\begin{array}{l}\mathtt{2}\ldots\mathtt{ebd}\underline{\mathtt{e}}\\ \mathtt{2}\ldots\mathtt{ebd}\underline{\mathtt{b}}\\ \mathtt{3}\ldots\mathtt{ebd}\underline{\mathtt{b}}\end{array}\right\}\rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{bccc\_} \\ \mathtt{1}\ldots\mathtt{ebd}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{2}\ldots\mathtt{bcdb\_}\\ \mathtt{5}\ldots\mathtt{ebd}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{5}\ldots\mathtt{bcca\_}\end{array}\right. \tag{476}$$

$$(302)\ \text{context}\ (\mathtt{bb},\mathtt{bc}) \tag{477}$$

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$$(229)\ \text{context}\ (\mathtt{a},\mathtt{c}) \tag{478}$$

$$\mathtt{5}\ldots\mathtt{d}\underline{\mathtt{d}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{bc\_} \Rightarrow \left.\begin{array}{l}\mathtt{3}\ldots\mathtt{dd}\underline{\mathtt{d}}\\ \mathtt{4}\ldots\mathtt{dd}\underline{\mathtt{d}}\end{array}\right\}\rightarrow\!\!\rightarrow \mathtt{4}\ldots\mathtt{caa} \tag{479}$$

$$\mathtt{5}\ldots\mathtt{b}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{cb\_} \Rightarrow \left.\begin{array}{l}\mathtt{3}\ldots\mathtt{bb}\underline{\mathtt{d}}\\ \mathtt{4}\ldots\mathtt{bb}\underline{\mathtt{d}}\end{array}\right\}\rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{abc\_} \tag{480}$$

$$\mathtt{4}\ldots\mathtt{b}\underline{\mathtt{d}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{aa\_} \Rightarrow \mathtt{1}\ldots\mathtt{bd}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{cbd\_} \tag{481}$$

$$\mathtt{5}\ldots\mathtt{b}\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{ab\_} \Rightarrow \left.\begin{array}{l}\mathtt{3}\ldots\mathtt{ba}\underline{\mathtt{d}}\\ \mathtt{4}\ldots\mathtt{ba}\underline{\mathtt{d}}\end{array}\right\}\rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{bca} \tag{482}$$

$$\mathtt{5}\ldots\beta\underline{\mathtt{a}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{cb\_} \Rightarrow \left.\begin{array}{l}\mathtt{3}\ldots\beta\mathtt{a}\underline{\mathtt{d}}\\ \mathtt{4}\ldots\beta\mathtt{a}\underline{\mathtt{d}}\end{array}\right\}\rightarrow\!\!\rightarrow \mathtt{1}\ldots\mathtt{abc\_}\ \text{for}\ \beta\in\{\mathtt{b},\mathtt{e}\} \tag{483}$$

$$\mathtt{5}\ldots\mathtt{b}\underline{\mathtt{b}} \rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{ab\_} \Rightarrow \left.\begin{array}{l}\mathtt{3}\ldots\mathtt{bb}\underline{\mathtt{d}}\\ \mathtt{4}\ldots\mathtt{bb}\underline{\mathtt{d}}\end{array}\right\}\rightarrow\!\!\rightarrow \mathtt{3}\ldots\mathtt{bca} \tag{484}$$

$$(333)\ \text{context}\ \begin{array}{l}(\mathtt{a},\mathtt{b})\\(\mathtt{d},\mathtt{b})\end{array} \tag{485}$$

$$(339)\ \text{context}\ (\mathtt{b},\mathtt{a}) \tag{486}$$

$$(342)\ \text{context}\ (\mathtt{d},\mathtt{c}) \tag{487}$$

$$\mathtt{4}\ldots\beta\underline{\mathtt{d}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bd}_{\_} \Rightarrow \left.\begin{array}{l}\mathtt{2}\ldots\beta\mathtt{d}\underline{\mathtt{b}}\\ \mathtt{3}\ldots\beta\mathtt{d}\underline{\mathtt{b}}\end{array}\right\} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{eca} \text{ for } \beta\in\{\mathtt{a},\mathtt{d}\} \tag{488}$$

$$(324) \text{ context } (\mathtt{b},\mathtt{a}) \tag{489}$$

$$(212) \text{ context } \begin{array}{l}(\mathtt{b},\mathtt{b})\\ (\mathtt{c},\mathtt{b})\\ (\mathtt{d},\mathtt{b})\end{array} \tag{490}$$

$$\mathtt{4}\ldots\mathtt{b}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bc}_{\_} \Rightarrow \left.\begin{array}{l}\mathtt{3}\ldots\mathtt{bd}\underline{\mathtt{d}}\\ \mathtt{4}\ldots\mathtt{bd}\underline{\mathtt{d}}\end{array}\right\} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{bab}_{\_} \tag{491}$$

$$(315) \text{ context } (\mathtt{b},\mathtt{a}) \tag{492}$$

$$(318) \text{ context } (\mathtt{d},\mathtt{c}) \tag{493}$$

$$\mathtt{3}\ldots\beta\underline{\mathtt{d}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bd}_{\_} \Rightarrow \left\{\begin{array}{l}\mathtt{4}\ldots\beta\mathtt{d}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{ecd}\\ \mathtt{2}\ldots\beta\mathtt{d}\underline{\mathtt{e}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{eca}\end{array}\right. \text{ for } \beta\in\{\mathtt{a},\mathtt{d}\} \tag{494}$$

$$\mathtt{3}\ldots\mathtt{b}\underline{\mathtt{d}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{aa}_{\_} \Rightarrow \mathtt{4}\ldots\mathtt{bd}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{cbc}_{\_} \tag{495}$$

$$\mathtt{3}\ldots\mathtt{b}\underline{\mathtt{c}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{cc}_{\_} \Rightarrow \left\{\begin{array}{l}\mathtt{3}\ldots\mathtt{ee}\underline{\mathtt{c}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{abc}_{\_}\\ \mathtt{1}\ldots\mathtt{ee}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{cdb}_{\_}\\ \mathtt{5}\ldots\mathtt{ee}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{5}\ldots\mathtt{cca}_{\_}\end{array}\right. \tag{496}$$

$$(302) \text{ context } \begin{array}{l}(\mathtt{b},\mathtt{c})\\ (\mathtt{e},\mathtt{c})\\ (\mathtt{b},\mathtt{a})\end{array} \tag{497}$$

$$\mathtt{3}\ldots\mathtt{e}\underline{\mathtt{c}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{cc}_{\_} \Rightarrow \mathtt{1}\ldots\mathtt{ec}\underline{\mathtt{c}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{abc}_{\_} \tag{498}$$

$$(308) \text{ context } \begin{array}{l}(\mathtt{a},\mathtt{b})\\ (\mathtt{d},\mathtt{b})\end{array} \tag{499}$$

$$\mathtt{3}\ldots\mathtt{b}\underline{\mathtt{c}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bc}_{\_} \Rightarrow \left\{\begin{array}{l}\mathtt{2}\ldots\mathtt{bc}\underline{\mathtt{e}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{caa}\\ \mathtt{3}\ldots\mathtt{ee}\underline{\mathtt{c}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{bdb}_{\_}\\ \mathtt{1}\ldots\mathtt{ee}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{bcb}_{\_}\\ \mathtt{5}\ldots\mathtt{ee}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{bcb}_{\_}\end{array}\right. . \tag{500}$$

$$\texttt{3}\ldots\texttt{b}\underline{\texttt{c}} \to\!\!\!\to \texttt{4}\ldots\texttt{ac\_} \Rightarrow \begin{cases} \texttt{1}\ldots\texttt{bc}\underline{\texttt{c}} \to\!\!\!\to \texttt{3}\ldots\texttt{dbc\_} \\ \texttt{3}\ldots\texttt{ee}\underline{\texttt{c}} \to\!\!\!\to \texttt{3}\ldots\texttt{dbc\_} \\ \texttt{1}\ldots\texttt{ee}\underline{\texttt{a}} \to\!\!\!\to \texttt{2}\ldots\texttt{adb\_} \\ \texttt{5}\ldots\texttt{ee}\underline{\texttt{a}} \to\!\!\!\to \texttt{5}\ldots\texttt{aca\_} \end{cases} \tag{501}$$

$$(206) \text{ for } \beta = \texttt{b} \text{ context } (\texttt{a}, \texttt{a}) \tag{502}$$

$$(204) \text{ context } (\texttt{a}, \texttt{b}) \tag{503}$$

$$\texttt{3}\ldots\texttt{a}\underline{\texttt{b}} \to\!\!\!\to \texttt{2}\ldots\texttt{ab\_} \Rightarrow \texttt{5}\ldots\texttt{ab}\underline{\texttt{b}} \to\!\!\!\to \texttt{3}\ldots\texttt{bcc} \tag{504}$$

$$\texttt{3}\ldots\texttt{d}\underline{\texttt{b}} \to\!\!\!\to \texttt{1}\ldots\texttt{bd\_} \Rightarrow \begin{cases} \texttt{4}\ldots\texttt{db}\underline{\texttt{a}} \to\!\!\!\to \texttt{3}\ldots\texttt{ecd} \\ \texttt{2}\ldots\texttt{db}\underline{\texttt{e}} \to\!\!\!\to \texttt{3}\ldots\texttt{eca} \end{cases} \tag{505}$$

$$(299) \text{ context } (\texttt{b}, \texttt{b}) \tag{506}$$

$$(290) \text{ context } \begin{array}{c} (\texttt{b}, \texttt{b}) \\ (\texttt{c}, \texttt{b}) \\ (\texttt{d}, \texttt{b}) \end{array} \tag{507}$$

$$\texttt{2}\ldots\texttt{b}\underline{\texttt{e}} \to\!\!\!\to \texttt{2}\ldots\texttt{ab\_} \Rightarrow \texttt{5}\ldots\texttt{be}\underline{\texttt{a}} \to\!\!\!\to \texttt{3}\ldots\texttt{bcc} \tag{508}$$

$$\texttt{2}\ldots\texttt{b}\underline{\texttt{e}} \to\!\!\!\to \texttt{2}\ldots\texttt{db\_} \Rightarrow \texttt{5}\ldots\texttt{be}\underline{\texttt{a}} \to\!\!\!\to \texttt{5}\ldots\texttt{ccc} \tag{509}$$

$$(296) \text{ context } (\texttt{d}, \texttt{b}) \tag{510}$$

$$(278) \text{ context } (\texttt{d}, \texttt{b}) \tag{511}$$

$$(194) \text{ context } (\texttt{a}, \texttt{a}) \tag{512}$$

$$\texttt{2}\ldots\texttt{a}\underline{\texttt{b}} \to\!\!\!\to \texttt{2}\ldots\texttt{ab\_} \Rightarrow \texttt{5}\ldots\texttt{ab}\underline{\texttt{a}} \to\!\!\!\to \texttt{3}\ldots\texttt{bcc} \tag{513}$$

$$(\mathbf{??}) \text{ for } \beta = \texttt{b} \text{ context } (\texttt{a}, \texttt{b}) \tag{514}$$

$$(285) \text{ context } \begin{array}{c} (\texttt{b}, \texttt{a}) \\ (\texttt{b}, \texttt{d}) \end{array} \tag{515}$$

$$(175) \text{ context } (\texttt{b}, \texttt{e}) \tag{516}$$

$$\mathtt{3\underline{a}b}\ldots \twoheadrightarrow \mathtt{2\_ae}\ldots \Rightarrow \left\{\begin{array}{l} \left.\begin{array}{c}\mathtt{5\underline{c}ab}\ldots \\ \mathtt{5\underline{e}ab}\ldots\end{array}\right\} \twoheadrightarrow \mathtt{5\_ccc}\ldots \\ \mathtt{4\underline{c}ab}\ldots \twoheadrightarrow \mathtt{3\_bcc}\ldots \end{array}\right. \tag{517}$$

$$\mathtt{3\underline{e}e}\ldots \twoheadrightarrow \mathtt{4\_ce}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{3\underline{e}ee}\ldots \twoheadrightarrow \mathtt{3bcb}\ldots \\ \mathtt{4\underline{c}ee}\ldots \twoheadrightarrow \mathtt{3\_bcc}\ldots \end{array}\right. \tag{518}$$

$$\mathtt{4\underline{b}b}\ldots \twoheadrightarrow \mathtt{4\_ce}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{4\underline{b}bb}\ldots \twoheadrightarrow \mathtt{3bcb}\ldots \\ \mathtt{3\underline{a}bb}\ldots \twoheadrightarrow \mathtt{3\_bcc}\ldots \end{array}\right. \tag{519}$$

$$\mathtt{4\underline{b}e}\ldots \twoheadrightarrow \mathtt{4\_cb}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{4\underline{b}be}\ldots \twoheadrightarrow \mathtt{2bab}\ldots \\ \mathtt{3\underline{a}be}\ldots \twoheadrightarrow \mathtt{3abc}\ldots \end{array}\right. \tag{520}$$

$$(258)\text{ context }(\mathtt{e},\mathtt{e}) \tag{521}$$

$$\mathtt{4\underline{c}e}\ldots \twoheadrightarrow \mathtt{2\_ae}\ldots \Rightarrow \mathtt{3\underline{a}ce}\ldots \twoheadrightarrow \mathtt{3\_bcc}\ldots \tag{522}$$

This needs a derivation with the pointer going to α first

$$\mathtt{4\underline{c}e}\ldots \twoheadrightarrow \mathtt{2\_ae}\ldots \Rightarrow \emptyset,\emptyset \tag{523}$$

$$\mathtt{5}\underline{\beta}\mathtt{e}\ldots \twoheadrightarrow \mathtt{3\_bc}\ldots \Rightarrow \mathtt{4\underline{e}}\beta\mathtt{e}\ldots \twoheadrightarrow \mathtt{5cba}\ldots \text{ for } \beta \in \{\mathtt{c},\mathtt{e}\} \tag{524}$$

$$\mathtt{1}\ldots\mathtt{b\underline{a}} \twoheadrightarrow \mathtt{1}\ldots\mathtt{ba\_} \Rightarrow \emptyset \tag{525}$$

$$\mathtt{1}\ldots\beta\mathtt{\underline{a}} \twoheadrightarrow \mathtt{2}\ldots\mathtt{db\_} \Rightarrow \emptyset \text{ for } \beta \in \{\mathtt{b},\mathtt{e}\} \tag{526}$$

$$\mathtt{1}\ldots\mathtt{a\underline{b}} \twoheadrightarrow \mathtt{2}\ldots\mathtt{db\_} \Rightarrow \emptyset \tag{527}$$

$$\mathtt{1}\ldots\beta\mathtt{\underline{c}} \twoheadrightarrow \mathtt{1}\ldots\mathtt{bc\_} \Rightarrow \emptyset \text{ for } \beta \in \{\mathtt{b},\mathtt{c},\mathtt{d}\} \tag{528}$$

$$\mathtt{1}\ldots\mathtt{b\underline{c}} \twoheadrightarrow \mathtt{4}\ldots\mathtt{cc\_} \Rightarrow \emptyset \tag{529}$$

$$\mathtt{3\underline{a}e}\ldots \twoheadrightarrow \mathtt{2\_ab}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{4\underline{b}eb}\ldots \twoheadrightarrow \mathtt{4\_cab}\ldots \\ \mathtt{3\underline{a}eb}\ldots \twoheadrightarrow \mathtt{3abc}\ldots \end{array}\right. \tag{530}$$

$$(179)\text{ context }\begin{array}{c}(\mathtt{a},\mathtt{a})\\(\mathtt{e},\mathtt{e})\end{array} \tag{531}$$

$$\mathtt{3\underline{e}a}\ldots \twoheadrightarrow \mathtt{4\_ca}\ldots \Rightarrow \left\{\begin{array}{l} \mathtt{3\underline{e}ea}\ldots \twoheadrightarrow \mathtt{4bcc}\ldots \\ \mathtt{4\underline{c}ea}\ldots \twoheadrightarrow \mathtt{1abc}\ldots \end{array}\right. \tag{532}$$

$$(173) \text{ context } (\mathtt{e},\mathtt{c}) \tag{533}$$

$$(173) \text{ context } (\mathtt{d},\mathtt{c}) \tag{534}$$

$$(168) \text{ context } (\beta,\mathtt{c}) \text{ for } \beta \in \{\mathtt{d},\mathtt{e}\} \tag{535}$$

$$\mathtt{3\underline{a}b}\ldots \rightarrow\rightarrow \mathtt{2\_ae}\ldots \Rightarrow \left\{ \begin{array}{l} \left.\begin{array}{c} \mathtt{5\underline{c}ab}\ldots \\ \mathtt{5\underline{e}ab}\ldots \end{array}\right\} \rightarrow\rightarrow \mathtt{5\_ccc}\ldots \\ \mathtt{4\underline{c}ab}\ldots \rightarrow\rightarrow \mathtt{3\_bcc}\ldots \end{array}\right. \tag{536}$$

$$\mathtt{1}\underline{\gamma}\beta\ldots \rightarrow\rightarrow \mathtt{4\_ca}\ldots \Rightarrow \mathtt{2\underline{a}}\gamma\beta \rightarrow\rightarrow \mathtt{1abc}\ldots \text{ for } \beta \in \{\mathtt{a},\mathtt{d}\} \text{ for } \gamma \in \{\mathtt{d},\mathtt{e}\} \tag{537}$$

$$\mathtt{1}\underline{\gamma}\mathtt{c}\ldots \rightarrow\rightarrow \mathtt{3\_ec}\ldots \Rightarrow \mathtt{2\underline{d}}\gamma\mathtt{c}\ldots \rightarrow\rightarrow \mathtt{3caa}\ldots \text{ for } \gamma \in \{\mathtt{d},\mathtt{e}\}. \tag{538}$$

$$\mathtt{1}\underline{\beta}\mathtt{d}\ldots \rightarrow\rightarrow \mathtt{4\_cb}\ldots \Rightarrow \mathtt{2\underline{a}}\beta\mathtt{d}\ldots \rightarrow\rightarrow \mathtt{3abc}\ldots \text{ for } \beta \in \{\mathtt{d},\mathtt{e}\} \tag{539}$$

$$(163) \text{ context } \begin{array}{c} (\mathtt{a},\mathtt{a}) \\ (\mathtt{d},\mathtt{a}) \\ (\mathtt{d},\mathtt{b}) \end{array} \tag{540}$$

$$(230) \text{ context } (\mathtt{c},\mathtt{c}) \tag{541}$$

8.5 The list of results that generate the IRR(5)

Applying the method again to the distinct right hand members of the above list of results gives the following that can be used to get all the members of IRR(5).

$$\mathtt{1}\underline{\beta}\ldots \rightarrow\rightarrow \mathtt{3\_e}\ldots \Rightarrow \left\{ \begin{array}{l} \mathtt{2\underline{a}}\beta\ldots \rightarrow\rightarrow \mathtt{2\_ae}\ldots \\ \mathtt{2\underline{c}}\beta\ldots \rightarrow\rightarrow \mathtt{5\_ce}\ldots \end{array}\right. \text{ for } \beta \in \{\mathtt{d},\mathtt{e}\} \text{ context } (\mathtt{aa},\mathtt{cc}) \tag{542}$$

(230)

$$\mathtt{1}\underline{\beta}\mathtt{aa}\ldots \rightarrow\rightarrow \mathtt{3\_ecc}\ldots \Rightarrow \mathtt{2\underline{d}}\beta\mathtt{aa}\ldots \rightarrow\rightarrow \mathtt{2cadb}\ldots \text{ for } \beta \in \{\mathtt{d},\mathtt{e}\} \tag{543}$$

$$\mathtt{1}\underline{\beta}\ldots \rightarrow\rightarrow \mathtt{4\_c}\ldots \Rightarrow \left\{ \begin{array}{l} \mathtt{2\underline{d}}\beta\ldots \rightarrow\rightarrow \mathtt{3\_bc}\ldots \\ \mathtt{2\underline{c}}\beta\ldots \rightarrow\rightarrow \mathtt{5\_cc}\ldots \end{array}\right. \text{ for } \beta \in \{\mathtt{d},\mathtt{e}\} \text{ context } (\mathtt{ca},\mathtt{bc}) \tag{544}$$

cf(163)(**??**)

$$\texttt{1}\underline{\texttt{d}}\texttt{aa}\ldots \rightarrow\rightarrow \texttt{3\_ecc}\ldots \Rightarrow \texttt{2}\underline{\texttt{d}}\texttt{daa}\ldots \rightarrow\rightarrow \texttt{2cadb}\ldots \tag{545}$$

$$\texttt{1}\underline{\texttt{e}}\texttt{aa}\ldots \rightarrow\rightarrow \texttt{3\_ecc}\ldots \Rightarrow \texttt{2}\underline{\texttt{d}}\texttt{eaa}\ldots \rightarrow\rightarrow \texttt{2cadb}\ldots \tag{546}$$

$$\texttt{1}\underline{\beta}\texttt{ca}\ldots \rightarrow\rightarrow \texttt{4\_cbc}\ldots \Rightarrow \texttt{2}\underline{\texttt{a}}\beta\texttt{ca}\ldots \rightarrow\rightarrow \texttt{2abab}\ldots \text{ for } \beta \in \{\texttt{d},\texttt{e}\} \tag{547}$$

$$\texttt{2}\underline{\beta}\ldots \rightarrow\rightarrow \texttt{2\_a}\ldots \Rightarrow \left.\begin{array}{c}\texttt{1}\underline{\texttt{d}}\beta\ldots\\ \texttt{1}\underline{\texttt{e}}\beta\ldots\end{array}\right\} \rightarrow\rightarrow \texttt{4\_ca}\ldots \text{ for } \beta \in \{\texttt{a},\texttt{d}\} \text{ context } \begin{array}{c}(\texttt{dc},\texttt{ec})\\ (\texttt{ec},\texttt{ec})\end{array} \tag{548}$$

cf(**??**)

$$\texttt{2}\underline{\texttt{a}}\texttt{d}\ldots \rightarrow\rightarrow \texttt{2\_ae}\ldots \Rightarrow \left.\begin{array}{c}\texttt{1}\underline{\texttt{d}}\texttt{da}\ldots\\ \texttt{1}\underline{\texttt{e}}\texttt{da}\ldots\end{array}\right\} \rightarrow\rightarrow \texttt{4\_cae}\ldots \text{ context } (\texttt{c},\texttt{c}) \tag{549}$$

$$\texttt{2}\underline{\texttt{a}}\texttt{ec}\ldots \rightarrow\rightarrow \texttt{2\_aec}\ldots \Rightarrow \left\{\begin{array}{l}\texttt{4}\underline{\texttt{b}}\texttt{e}\gamma\texttt{d}\ldots \rightarrow\rightarrow \texttt{4\_caec}\ldots\\ \texttt{3}\underline{\texttt{a}}\texttt{e}\gamma\texttt{d}\ldots \rightarrow\rightarrow \texttt{3\_bccc}\ldots\end{array}\right. \text{ for } \gamma \in \{\texttt{c},\texttt{e}\} \tag{550}$$

$$\texttt{2}\underline{\texttt{c}}\ldots \rightarrow\rightarrow \texttt{5\_c}\ldots \Rightarrow \left.\begin{array}{c}\texttt{1}\underline{\texttt{d}}\texttt{c}\ldots\\ \texttt{1}\underline{\texttt{e}}\texttt{c}\ldots\end{array}\right\} \rightarrow\rightarrow \texttt{3\_ec}\ldots \text{ context } \begin{array}{c}(\texttt{dc},\texttt{ec})\\ (\texttt{ec},\texttt{ec})\end{array} \tag{551}$$

cf(168)

$$\texttt{2}\underline{\texttt{c}}\texttt{d}\ldots \rightarrow\rightarrow \texttt{5\_ce}\ldots \Rightarrow \left\{\begin{array}{l}\left.\begin{array}{c}\texttt{5}\underline{\texttt{c}}\texttt{ad}\ldots\\ \texttt{5}\underline{\texttt{e}}\texttt{ad}\ldots\\ \texttt{4}\underline{\texttt{e}}\texttt{ec}\ldots\\ \texttt{4}\underline{\texttt{e}}\texttt{cc}\ldots\end{array}\right\} \rightarrow\rightarrow \texttt{2\_ece}\ldots\\ \left.\begin{array}{c}\texttt{1}\underline{\texttt{d}}\texttt{aa}\ldots\\ \texttt{1}\underline{\texttt{e}}\texttt{aa}\ldots\\ \texttt{4}\underline{\texttt{b}}\texttt{cd}\ldots\\ \texttt{1}\underline{\texttt{d}}\texttt{ab}\ldots\\ \texttt{1}\underline{\texttt{e}}\texttt{ab}\ldots\\ \texttt{3}\underline{\texttt{e}}\texttt{ad}\ldots\end{array}\right\} \rightarrow\rightarrow \texttt{3\_ece}\ldots\\ \left.\begin{array}{c}\texttt{3}\underline{\texttt{a}}\texttt{cd}\ldots\\ \texttt{4}\underline{\texttt{c}}\texttt{ad}\ldots\end{array}\right\} \rightarrow\rightarrow \texttt{5\_ccc}\ldots\end{array}\right. \text{ context } (\texttt{c},\texttt{c}) \tag{552}$$

$$\texttt{3}\underline{\texttt{a}}\ldots \rightarrow\rightarrow \texttt{2\_a}\ldots \Rightarrow \texttt{3}\underline{\texttt{e}}\texttt{a}\ldots \rightarrow\rightarrow \texttt{4\_ca}\ldots \text{ context } (\texttt{cc},\texttt{bc}) \tag{553}$$

cf(175)(391)

$$\mathtt{3\underline{a}\ldots} \to\!\!\to \mathtt{3\_b\ldots} \Rightarrow \begin{cases} \mathtt{3\underline{e}a\ldots} \to\!\!\to \mathtt{4\_cb\ldots} \\ \mathtt{4\underline{c}a\ldots} \to\!\!\to \mathtt{2\_ab\ldots} \end{cases} \text{context } (\mathtt{ce}, \mathtt{cc}) \tag{554}$$

$$\mathtt{3\underline{a}cc\ldots} \to\!\!\to \mathtt{2\_abc\ldots} \Rightarrow \left\{ \begin{array}{l} \left. \begin{array}{l} \mathtt{5\underline{c}acc\ldots} \\ \mathtt{5\underline{e}acc\ldots} \end{array} \right\} \to\!\!\to \mathtt{2dbab\ldots} \\ \mathtt{4\underline{c}acc\ldots} \to\!\!\to \mathtt{3ccbc\ldots} \end{array} \right. \tag{555}$$

$$\mathtt{3\underline{a}ce\ldots} \to\!\!\to \mathtt{3\_bcc\ldots} \Rightarrow \left. \begin{array}{l} \mathtt{5\underline{c}ace\ldots} \\ \mathtt{5\underline{e}ace\ldots} \end{array} \right\} \to\!\!\to \mathtt{2cbab\ldots} \tag{556}$$

$$\mathtt{4\underline{b}c\ldots} \to\!\!\to \mathtt{4\_cb\ldots} \Rightarrow \left\{ \begin{array}{l} \left. \begin{array}{l} \mathtt{5\underline{c}eb\ldots} \\ \mathtt{5\underline{e}eb\ldots} \\ \mathtt{2\underline{d}db\ldots} \\ \mathtt{2\underline{d}eb\ldots} \end{array} \right\} \to\!\!\to \mathtt{3\_bcb\ldots} \\ \left. \begin{array}{l} \mathtt{2\underline{c}db\ldots} \\ \mathtt{2\underline{c}eb\ldots} \end{array} \right\} \to\!\!\to \mathtt{5\_ccb\ldots} \end{array} \right. \quad \text{context} \begin{array}{l} (\mathtt{b}, \mathtt{c}) \\ (\mathtt{c}, \mathtt{c}) \end{array} \tag{557}$$

$$\mathtt{4\underline{b}c}\beta\mathtt{\ldots} \to\!\!\to \mathtt{4\_cbc\ldots} \Rightarrow \left\{ \begin{array}{l} \left. \begin{array}{l} \mathtt{4\underline{b}bc}\beta\mathtt{\ldots} \\ \mathtt{4\underline{e}eb}\beta\mathtt{\ldots} \end{array} \right\} \to\!\!\to \mathtt{1babd\ldots} \\ \left. \begin{array}{l} \mathtt{3\underline{a}bc}\beta\mathtt{\ldots} \\ \mathtt{2\underline{a}db}\beta\mathtt{\ldots} \\ \mathtt{2\underline{a}eb}\beta\mathtt{\ldots} \\ \mathtt{4\underline{c}eb}\beta\mathtt{\ldots} \end{array} \right\} \to\!\!\to \mathtt{2abab\ldots} \end{array} \right. \quad \text{for } \beta \in \{\mathtt{b}, \mathtt{c}\} \tag{558}$$

$$\mathtt{4\underline{b}cb\ldots} \to\!\!\to \mathtt{4\_cbc\ldots} \Rightarrow \begin{cases} \mathtt{4\underline{b}bba\ldots} \to\!\!\to \mathtt{1babd\ldots} \\ \mathtt{3\underline{a}bba\ldots} \to\!\!\to \mathtt{2abab\ldots} \end{cases} \tag{559}$$

$$\mathtt{4\underline{b}c\ldots} \to\!\!\to \mathtt{4\_ca\ldots} \Rightarrow \left\{ \begin{array}{l} \left. \begin{array}{l} \mathtt{2\underline{d}}\beta\mathtt{b\ldots} \\ \mathtt{5\underline{c}eb\ldots} \\ \mathtt{5\underline{e}eb\ldots} \end{array} \right\} \to\!\!\to \mathtt{3\_bca\ldots} \\ \mathtt{2\underline{c}}\beta\mathtt{b\ldots} \to\!\!\to \mathtt{5\_cca\ldots} \end{array} \right. \quad \text{for } \beta \in \{\mathtt{d}, \mathtt{e}\} \text{ context } (\mathtt{e}, \mathtt{e}) \tag{560}$$

$$\mathtt{4\underline{b}ce\ldots} \to\!\!\to \mathtt{4\_cae\ldots} \Rightarrow \left\{ \begin{array}{l} \left. \begin{array}{l} \mathtt{4\underline{b}bce\ldots} \\ \mathtt{3\underline{e}ebe\ldots} \end{array} \right\} \to\!\!\to \mathtt{5bcca\ldots} \\ \left. \begin{array}{l} \mathtt{3\underline{a}bce\ldots} \\ \mathtt{4\underline{c}ebe\ldots} \\ \mathtt{2\underline{a}}\beta\mathtt{be\ldots} \end{array} \right\} \to\!\!\to \mathtt{2abcb\ldots} \end{array} \right. \quad \text{for } \beta \in \{\mathtt{d}, \mathtt{e}\} \tag{561}$$

$$\texttt{4}\underline{\texttt{b}}\texttt{eb}\ldots \rightarrow\rightarrow \texttt{4\_cab}\ldots \Rightarrow \begin{cases} \texttt{4}\underline{\texttt{b}}\texttt{beb}\ldots \rightarrow\rightarrow \texttt{4bccb}\ldots \\ \texttt{3}\underline{\texttt{a}}\texttt{beb}\ldots \rightarrow\rightarrow \texttt{2abdb}\ldots \end{cases} \tag{562}$$

$$\texttt{4}\underline{\texttt{e}}\ldots \rightarrow\rightarrow \texttt{2\_e}\ldots \Rightarrow \texttt{4}\underline{\texttt{b}}\texttt{e}\ldots \rightarrow\rightarrow \texttt{4\_ce}\ldots \ \text{context} \begin{matrix} (\texttt{cc},\texttt{cc}) \\ (\texttt{ec},\texttt{cc}) \end{matrix} \tag{563}$$

$$\texttt{2}\ldots\underline{\texttt{b}} \rightarrow\rightarrow \texttt{1}\ldots\texttt{d\_} \Rightarrow \begin{cases} \texttt{3}\ldots\texttt{b}\underline{\texttt{c}} \rightarrow\rightarrow \texttt{2}\ldots\texttt{ab\_} \\ \texttt{1}\ldots\texttt{b}\underline{\texttt{a}} \rightarrow\rightarrow \texttt{2}\ldots\texttt{db\_} \\ \texttt{5}\ldots\texttt{b}\underline{\texttt{a}} \rightarrow\rightarrow \texttt{2}\ldots\texttt{db\_} \end{cases} \text{context } (\texttt{ba},\texttt{ab}) \tag{564}$$

$$\texttt{2}\ldots\underline{\texttt{b}} \rightarrow\rightarrow \texttt{2}\ldots\texttt{b\_} \Rightarrow \begin{cases} \texttt{3}\ldots\texttt{b}\underline{\texttt{c}} \rightarrow\rightarrow \texttt{1}\ldots\texttt{bc\_} \\ \texttt{1}\ldots\texttt{b}\underline{\texttt{a}} \rightarrow\rightarrow \texttt{1}\ldots\texttt{ba\_} \end{cases} \text{context} \begin{matrix} (\texttt{bd},\texttt{ad}) \\ (\texttt{dd},\texttt{ca}) \end{matrix} \tag{565}$$

$$\texttt{2}\ldots\texttt{bd}\underline{\texttt{b}} \rightarrow\rightarrow \texttt{2}\ldots\texttt{adb\_} \Rightarrow \texttt{5}\ldots\texttt{bdb}\underline{\texttt{a}} \rightarrow\rightarrow \texttt{2}\ldots\texttt{eccc} \tag{566}$$

$$\texttt{2}\ldots\texttt{dd}\underline{\texttt{b}} \rightarrow\rightarrow \texttt{2}\ldots\texttt{cab\_} \Rightarrow \texttt{5}\ldots\texttt{ddb}\underline{\texttt{a}} \rightarrow\rightarrow \texttt{2}\ldots\texttt{abcc} \tag{567}$$

$$\texttt{2}\ldots\underline{\texttt{e}} \rightarrow\rightarrow \texttt{2}\ldots\texttt{b\_} \Rightarrow \begin{cases} \texttt{3}\ldots\texttt{e}\underline{\texttt{c}} \rightarrow\rightarrow \texttt{1}\ldots\texttt{bc\_} \\ \texttt{1}\ldots\texttt{e}\underline{\texttt{a}} \rightarrow\rightarrow \texttt{1}\ldots\texttt{ba\_} \end{cases} \text{context} \begin{matrix} (\texttt{bd},\texttt{ad}) \\ (\texttt{dd},\texttt{ca}) \end{matrix} \tag{568}$$

$$\texttt{2}\ldots\texttt{bd}\underline{\texttt{e}} \rightarrow\rightarrow \texttt{2}\ldots\texttt{adb\_} \Rightarrow \texttt{5}\ldots\texttt{bde}\underline{\texttt{a}} \rightarrow\rightarrow \texttt{2}\ldots\texttt{eccc} \tag{569}$$

$$\texttt{2}\ldots\underline{\texttt{e}} \rightarrow\rightarrow \texttt{3}\ldots\texttt{c\_} \Rightarrow \begin{cases} \texttt{3}\ldots\texttt{e}\underline{\texttt{c}} \rightarrow\rightarrow \texttt{4}\ldots\texttt{cc\_} \\ \texttt{1}\ldots\texttt{e}\underline{\texttt{a}} \rightarrow\rightarrow \texttt{2}\ldots\texttt{db\_} \\ \texttt{5}\ldots\texttt{e}\underline{\texttt{a}} \rightarrow\rightarrow \texttt{3}\ldots\texttt{cb\_} \end{cases} \text{context} \begin{matrix} (\texttt{bc},\texttt{cc}) \\ (\texttt{ec},\texttt{cc}) \\ (\texttt{bc},\texttt{ac}) \end{matrix} \tag{570}$$

(??)

$$\texttt{3}\ldots\underline{\texttt{b}} \rightarrow\rightarrow \texttt{1}\ldots\texttt{d\_} \Rightarrow \begin{cases} \texttt{1}\ldots\texttt{b}\underline{\texttt{c}} \rightarrow\rightarrow \texttt{2}\ldots\texttt{ab\_} \\ \texttt{5}\ldots\texttt{b}\underline{\texttt{b}} \rightarrow\rightarrow \texttt{2}\ldots\texttt{db\_} \end{cases} \text{context } (\texttt{ba},\texttt{ab}) \tag{571}$$

$$\texttt{3}\ldots\underline{\beta} \rightarrow\rightarrow \texttt{2}\ldots\texttt{b\_} \Rightarrow \begin{cases} \texttt{1}\ldots\beta\underline{\texttt{c}} \rightarrow\rightarrow \texttt{1}\ldots\texttt{bc\_} \\ \texttt{4}\ldots\beta\underline{\texttt{a}} \rightarrow\rightarrow \texttt{3}\ldots\texttt{bc\_} \\ \texttt{2}\ldots\beta\underline{\texttt{e}} \rightarrow\rightarrow \texttt{1}\ldots\texttt{bd\_} \end{cases} \text{context} \begin{matrix} (\texttt{bd},\texttt{ad}) \\ (\texttt{dd},\texttt{ca}) \end{matrix} \text{ for } \beta \in \{\texttt{b},\texttt{c},\texttt{d}\} \tag{572}$$

only $\beta = \mathtt{b}$ needed so far

$$\mathtt{3\ldots bd\underline{b}} \rightarrow\!\!\rightarrow \mathtt{2\ldots adb\_} \Rightarrow \mathtt{5\ldots bdb\underline{b}} \rightarrow\!\!\rightarrow \mathtt{2\ldots eccc} \tag{573}$$

$$\mathtt{3\ldots dd\underline{b}} \rightarrow\!\!\rightarrow \mathtt{2\ldots cab\_} \Rightarrow \mathtt{5\ldots ddb\underline{b}} \rightarrow\!\!\rightarrow \mathtt{2\ldots abcc} \tag{574}$$

$$\mathtt{3\ldots \underline{c}} \rightarrow\!\!\rightarrow \mathtt{1\ldots c\_} \Rightarrow \begin{cases} \mathtt{1\ldots c\underline{c}} \rightarrow\!\!\rightarrow \mathtt{2\ldots db\_} \\ \mathtt{4\ldots c\underline{a}} \rightarrow\!\!\rightarrow \mathtt{2\ldots db\_} \\ \mathtt{5\ldots c\underline{b}} \rightarrow\!\!\rightarrow \mathtt{2\ldots cb\_} \end{cases} \text{context } (\mathtt{be}, \mathtt{ab}) \tag{575}$$

$$\mathtt{3\ldots be\underline{c}} \rightarrow\!\!\rightarrow \mathtt{1\ldots abc\_} \Rightarrow \mathtt{2\ldots bec\underline{e}} \rightarrow\!\!\rightarrow \mathtt{3\ldots bcaa} \tag{576}$$

$$\mathtt{3\ldots b\underline{c}} \rightarrow\!\!\rightarrow \mathtt{2\ldots ab\_} \Rightarrow \begin{cases} \mathtt{1\ldots ee\underline{c}} \rightarrow\!\!\rightarrow \mathtt{1\ldots abc\_} \\ \mathtt{1\ldots ee\underline{a}} \rightarrow\!\!\rightarrow \mathtt{1\ldots aba\_} \end{cases} \text{context } (\mathtt{d}, \mathtt{b}) \tag{577}$$

$$\mathtt{3\ldots db\underline{c}} \rightarrow\!\!\rightarrow \mathtt{2\ldots bab\_} \Rightarrow \left. \begin{array}{l} \mathtt{5\ldots dee\underline{a}} \\ \mathtt{5\ldots dbc\underline{b}} \end{array} \right\} \rightarrow\!\!\rightarrow \mathtt{4\ldots cbcc} \tag{578}$$

$$\mathtt{3\ldots \underline{c}} \rightarrow\!\!\rightarrow \mathtt{2\ldots b\_} \Rightarrow \begin{cases} \mathtt{1\ldots c\underline{c}} \rightarrow\!\!\rightarrow \mathtt{1\ldots bc\_} \\ \mathtt{4\ldots c\underline{a}} \rightarrow\!\!\rightarrow \mathtt{3\ldots bc\_} \\ \mathtt{2\ldots c\underline{e}} \rightarrow\!\!\rightarrow \mathtt{1\ldots bd\_} \end{cases} \text{context } \begin{array}{c} (\mathtt{be}, \mathtt{ba}) \\ (\mathtt{ce}, \mathtt{ba}) \\ (\mathtt{de}, \mathtt{ba}) \\ (\mathtt{db}, \mathtt{ba}) \\ (\mathtt{ee}, \mathtt{bd}) \end{array} \tag{579}$$

$$\mathtt{3\ldots}\beta\mathtt{e\underline{c}} \rightarrow\!\!\rightarrow \mathtt{2\ldots bab\_} \Rightarrow \mathtt{5\ldots}\beta\mathtt{ec\underline{b}} \rightarrow\!\!\rightarrow \mathtt{4\ldots cbcc} \text{ for } \beta \in \{\mathtt{b}, \mathtt{c}, \mathtt{d}\} \tag{580}$$

$$\mathtt{3\ldots ce\underline{c}} \rightarrow\!\!\rightarrow \mathtt{2\ldots bdb\_} \Rightarrow \mathtt{5\ldots eec\underline{b}} \rightarrow\!\!\rightarrow \mathtt{3\ldots eccc} \tag{581}$$

$$\mathtt{3\ldots ee\underline{c}} \rightarrow\!\!\rightarrow \mathtt{2\ldots bdb\_} \Rightarrow \mathtt{5\ldots eec\underline{b}} \rightarrow\!\!\rightarrow \mathtt{3\ldots eccc} \tag{582}$$

$$\mathtt{3\ldots \underline{c}} \rightarrow\!\!\rightarrow \mathtt{3\ldots c\_} \Rightarrow \begin{cases} \mathtt{1\ldots c\underline{c}} \rightarrow\!\!\rightarrow \mathtt{4\ldots cc\_} \\ \mathtt{4\ldots c\underline{a}} \rightarrow\!\!\rightarrow \mathtt{2\ldots ab\_} \\ \mathtt{2\ldots c\underline{e}} \rightarrow\!\!\rightarrow \mathtt{2\ldots ab\_} \\ \mathtt{5\ldots c\underline{b}} \rightarrow\!\!\rightarrow \mathtt{3\ldots cb\_} \end{cases} \text{context } \begin{array}{c} (\mathtt{ee}, \mathtt{db}) \\ (\mathtt{ee}, \mathtt{ab}) \end{array} \tag{583}$$

$$\mathtt{3\ldots \underline{d}} \rightarrow\!\!\rightarrow \mathtt{1\ldots d\_} \Rightarrow \begin{cases} \mathtt{1\ldots d\underline{c}} \rightarrow\!\!\rightarrow \mathtt{2\ldots ab\_} \\ \mathtt{5\ldots d\underline{b}} \rightarrow\!\!\rightarrow \mathtt{2\ldots db\_} \end{cases} \text{context } (\mathtt{bd}, \mathtt{ab}) \tag{584}$$

$$\mathtt{3}\ldots\underline{\mathtt{d}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{b\_} \Rightarrow \left\{\begin{array}{l} \mathtt{1}\ldots\mathtt{d}\underline{\mathtt{c}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bc\_} \\ \mathtt{4}\ldots\mathtt{d}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bc\_} \\ \mathtt{2}\ldots\mathtt{d}\underline{\mathtt{e}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bd\_} \end{array}\right. \text{context } (\mathtt{bd},\mathtt{ba}) \tag{585}$$

$$\mathtt{3}\ldots\underline{\mathtt{d}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{c\_} \Rightarrow \left\{\begin{array}{l} \mathtt{1}\ldots\mathtt{d}\underline{\mathtt{c}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{db\_} \\ \mathtt{4}\ldots\mathtt{d}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{db\_} \\ \mathtt{5}\ldots\mathtt{d}\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{cb\_} \end{array}\right. \text{context } \begin{array}{c} (\mathtt{ba},\mathtt{ab}) \\ (\mathtt{ea},\mathtt{ab}) \end{array} \tag{586}$$

$$\mathtt{3}\ldots\gamma\mathtt{a}\underline{\mathtt{d}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{abc\_} \Rightarrow \mathtt{2}\ldots\gamma\mathtt{ad}\underline{\mathtt{e}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bcaa} \text{ for } \gamma \in \{\mathtt{b},\mathtt{e}\} \tag{587}$$

$$\mathtt{3}\ldots\mathtt{bd}\underline{\mathtt{d}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{bab\_} \Rightarrow \mathtt{5}\ldots\mathtt{bdd}\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{cbcc} \tag{588}$$

$$\mathtt{4}\ldots\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{b\_} \Rightarrow \left\{\begin{array}{l} \mathtt{5}\ldots\mathtt{a}\underline{\mathtt{d}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bc\_} \\ \left.\begin{array}{l} \mathtt{2}\ldots\mathtt{a}\underline{\mathtt{b}} \\ \mathtt{3}\ldots\mathtt{a}\underline{\mathtt{b}} \end{array}\right\} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{bd\_} \\ \mathtt{1}\ldots\mathtt{a}\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{1}\ldots\mathtt{ba\_} \end{array}\right. \text{context } \begin{array}{c} (\mathtt{bc},\mathtt{bd}) \\ (\mathtt{dd},\mathtt{ca}) \end{array} \tag{589}$$

$$\mathtt{4}\ldots\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{b\_} \Rightarrow \left.\begin{array}{l} \mathtt{2}\ldots\mathtt{a}\underline{\mathtt{b}} \\ \mathtt{3}\ldots\mathtt{a}\underline{\mathtt{b}} \end{array}\right\} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bc\_} \text{ context } \begin{array}{c} (\mathtt{bc},\mathtt{cc}) \\ (\mathtt{ec},\mathtt{cc}) \\ (\mathtt{bc},\mathtt{ac}) \end{array} \tag{590}$$

$$\mathtt{4}\ldots\mathtt{c}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{cb\_} \Rightarrow \mathtt{5}\ldots\mathtt{ca}\underline{\mathtt{d}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{abc\_} \text{ context } \begin{array}{c} (\mathtt{b},\mathtt{c}) \\ (\mathtt{e},\mathtt{c}) \\ (\mathtt{b},\mathtt{a}) \end{array} \tag{591}$$

$$\mathtt{4}\ldots\mathtt{bc}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{ccb\_} \Rightarrow \mathtt{1}\ldots\mathtt{bca}\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bccc} \tag{592}$$

$$\mathtt{4}\ldots\mathtt{ec}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{ccb\_} \Rightarrow \mathtt{1}\ldots\mathtt{eca}\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{bccc} \tag{593}$$

$$\mathtt{4}\ldots\mathtt{bc}\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{acb\_} \Rightarrow \mathtt{1}\ldots\mathtt{bca}\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{5}\ldots\mathtt{cccc} \tag{594}$$

$$\mathtt{4}\ldots\underline{\mathtt{a}} \rightarrow\!\!\!\rightarrow \mathtt{3}\ldots\mathtt{c\_} \Rightarrow \left\{\begin{array}{l} \mathtt{5}\ldots\mathtt{a}\underline{\mathtt{d}} \rightarrow\!\!\!\rightarrow \mathtt{4}\ldots\mathtt{cc\_} \\ \left.\begin{array}{l} \mathtt{2}\ldots\mathtt{a}\underline{\mathtt{b}} \\ \mathtt{3}\ldots\mathtt{a}\underline{\mathtt{b}} \end{array}\right\} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{ab\_} \\ \mathtt{1}\ldots\mathtt{a}\underline{\mathtt{b}} \rightarrow\!\!\!\rightarrow \mathtt{2}\ldots\mathtt{db\_} \end{array}\right. \text{context } (\mathtt{bd},\mathtt{cb}) \tag{595}$$

$$4\ldots\underline{\mathtt{d}} \to\to 1\ldots\mathtt{d}\_ \Rightarrow \left\{\begin{array}{l} 5\ldots\mathtt{d}\underline{\mathtt{d}} \to\to 2\ldots\mathtt{ab}\_ \\ 1\ldots\mathtt{d}\underline{\mathtt{b}} \to\to 2\ldots\mathtt{db}\_ \end{array}\right. \quad \text{context } (\mathtt{bd},\mathtt{ab}) \tag{596}$$

$$4\ldots\underline{\mathtt{d}} \to\to 2\ldots\mathtt{b}\_ \Rightarrow \left\{\begin{array}{l} 5\ldots\mathtt{d}\underline{\mathtt{d}} \to\to 1\ldots\mathtt{bc}\_ \\ \left.\begin{array}{l} 2\ldots\mathtt{d}\underline{\mathtt{b}} \\ 3\ldots\mathtt{d}\underline{\mathtt{b}} \end{array}\right\} \to\to 1\ldots\mathtt{bd}\_ \\ 1\ldots\mathtt{d}\underline{\mathtt{b}} \to\to 1\ldots\mathtt{ba}\_ \end{array}\right. \quad \text{context } (\mathtt{bd},\mathtt{ba}) \tag{597}$$

(216)

$$5\ldots\underline{\mathtt{a}} \to\to 2\ldots\mathtt{b}\_ \Rightarrow \left.\begin{array}{l} 3\ldots\mathtt{a}\underline{\mathtt{d}} \\ 4\ldots\mathtt{a}\underline{\mathtt{d}} \end{array}\right\} \to\to 1\ldots\mathtt{bd}\_ \text{ context } \begin{array}{c} (\mathtt{ee},\mathtt{bc}) \\ (\mathtt{be},\mathtt{bd}) \end{array} \tag{598}$$

$$5\ldots\mathtt{db}\underline{\mathtt{a}} \to\to 2\ldots\mathtt{bdb}\_ \Rightarrow \left.\begin{array}{l} 3\ldots\mathtt{dba}\underline{\mathtt{d}} \\ 4\ldots\mathtt{dba}\underline{\mathtt{d}} \end{array}\right\} \to\to 1\ldots\mathtt{bdbd}\_ \tag{599}$$

$$5\ldots\underline{\mathtt{a}} \to\to 5\ldots\mathtt{a}\_ \Rightarrow \left.\begin{array}{l} 3\ldots\mathtt{a}\underline{\mathtt{d}} \\ 4\ldots\mathtt{a}\underline{\mathtt{d}} \end{array}\right\} \to\to 3\ldots\mathtt{aa}\_ \text{ context } \begin{array}{c} (\mathtt{ee},\mathtt{ac}) \\ (\mathtt{ee},\mathtt{cc}) \end{array} \tag{600}$$

$$5\ldots\underline{\mathtt{b}} \to\to 2\ldots\mathtt{b}\_ \Rightarrow \left.\begin{array}{l} 3\ldots\mathtt{b}\underline{\mathtt{d}} \\ 4\ldots\mathtt{b}\underline{\mathtt{d}} \end{array}\right\} \to\to 1\ldots\mathtt{bd}\_ \text{ context } \begin{array}{c} (\mathtt{bc},\mathtt{bc}) \\ (\mathtt{db},\mathtt{bd}) \\ (\mathtt{ad},\mathtt{bd}) \\ (\mathtt{dd},\mathtt{bd}) \end{array} \tag{601}$$

$$5\ldots\mathtt{bd}\underline{\mathtt{b}} \to\to 3\ldots\mathtt{aab}\_ \Rightarrow \left.\begin{array}{l} 3\ldots\mathtt{bdb}\underline{\mathtt{d}} \\ 4\ldots\mathtt{bdb}\underline{\mathtt{d}} \end{array}\right\} \to\to 4\ldots\mathtt{cbcc}\_ \tag{602}$$

$$5\ldots\mathtt{d}\underline{\mathtt{b}} \to\to 3\ldots\mathtt{cb}\_ \Rightarrow \left.\begin{array}{l} 3\ldots\mathtt{db}\underline{\mathtt{d}} \\ 4\ldots\mathtt{db}\underline{\mathtt{d}} \end{array}\right\} \to\to 1\ldots\mathtt{abc}\_ \text{ context } (\mathtt{c},\mathtt{c}) \tag{603}$$

$$5\ldots\underline{\mathtt{b}} \to\to 5\ldots\mathtt{a}\_ \Rightarrow \left.\begin{array}{l} 3\ldots\mathtt{b}\underline{\mathtt{d}} \\ 4\ldots\mathtt{b}\underline{\mathtt{d}} \end{array}\right\} \to\to 3\ldots\mathtt{aa}\_ \text{ context } \begin{array}{c} (\mathtt{bc},\mathtt{cc}) \\ (\mathtt{ec},\mathtt{cc}) \\ (\mathtt{bc},\mathtt{ac}) \end{array} \tag{604}$$

$$5\ldots\mathtt{ba}\underline{\mathtt{d}} \to\to 1\ldots\mathtt{abc}\_ \Rightarrow \left.\begin{array}{l} 3\ldots\mathtt{bad}\underline{\mathtt{d}} \\ 4\ldots\mathtt{bad}\underline{\mathtt{d}} \end{array}\right\} \to\to 3\ldots\mathtt{bcaa} \tag{605}$$

$$5\ldots\underline{\beta} \to\to 2\ldots\mathtt{b}\_ \Rightarrow \left.\begin{array}{l} 3\ldots\beta\underline{\mathtt{d}} \\ 4\ldots\beta\underline{\mathtt{d}} \end{array}\right\} \to\to 1\ldots\mathtt{bd}\_ \text{ for } \beta\in\{\mathtt{a},\mathtt{d}\} \text{ context } \begin{array}{c} (\mathtt{ad},\mathtt{ba}) \\ (\mathtt{dd},\mathtt{ba}) \end{array} \tag{606}$$

(219)

$$\mathtt{5}\ldots\underline{\mathtt{d}} \rightarrow\rightarrow \mathtt{4}\ldots\mathtt{c}_{-} \Rightarrow \left.\begin{array}{c}\mathtt{3}\ldots\mathtt{d}\underline{\mathtt{d}}\\ \mathtt{4}\ldots\mathtt{d}\underline{\mathtt{d}}\end{array}\right\} \rightarrow\rightarrow \mathtt{3}\ldots\mathtt{cc}_{-}\ \text{context}\ \begin{array}{c}(\mathtt{bd},\mathtt{aa})\\ (\mathtt{dd},\mathtt{cc})\end{array} \qquad (607)$$

(229)

9 Counting the IRR(n) for each n derived from Table 2

`IRR(2-10): 55,78,163,291,702,1578,3958,9686,24631`

Much of the above description treats an IRR is a triplet $\mathtt{A} \to \mathtt{B} \to \mathtt{C}$ that might be abbreviated to get patterns that match many triplets. However IRR's in general are defined uniquely by the CS $\mathtt{B}$ and from it $\mathtt{C}$ is uniquely defined, but in general there are a set of CS's $\mathtt{A}$ forming the origin of $\mathtt{B}$ i.e. such that $\mathtt{A} \in \mathtt{O}_{\mathtt{1}}(\mathtt{B})$. Thus many distinct triplets can be part of the same IRR with the same $\mathtt{B}$ and $\mathtt{C}$. These definitions work because if an IRR is a union of a set of triplets with the same $\mathtt{B}$ and $\mathtt{C}$, $\mathtt{F}$ applied to the IRR is the union of $\mathtt{F}$ applied to each individual triplet. This fact is obvious from the arguments leading up to Theorem 2.2 and it was taken for granted in Tables 1 and 2 where the IRR outlines have been represented with a single origin in each case. Therefore for counting purposes, the results need to be grouped together with a common $\mathtt{B}$ and $\mathtt{C}$. To do this the key fact seems to be the following regardless of the number of origins of the IRR: from any IRR $\mathtt{X}$ of length $\mathtt{n}$ and symbol α, a distinct IRR $\mathtt{F}(\mathtt{X})$ of length $\mathtt{n}+\mathtt{1}$ is produced if the backward search leads to the reachability of the LHS (the middle member of an IRR triplet which does not usually have to mentioned explicitly) for that value of α. Otherwise no $\mathtt{F}(\mathtt{X})$ is produced.

From this it follows that if in a set $\mathtt{S}$ of two or more IRR($\mathtt{n}$) outlines, the RHS is the same for each, then they in general represent a set of $\mathtt{r} \in \text{IRR}(\mathtt{n})$ of which some might have at least two origins. Then for each such $\mathtt{r}$ the set of α values is the union of the sets of α values for each IRR outline in $\mathtt{S}$ (see results in Table 2 matching $\mathtt{r}$), and this number is the number of IRR($\mathtt{n}+\mathtt{1}$) derived from $\mathtt{r}$ according to Theorem 2.2. This can be used to establish sets of IRR outlines that represent the same IRR of types RL and RR (types LR and LL will be considered in the next section) with multiple origins, starting from the set of IRR(3) i.e. (36) to (81) and to define inductively the set of all IRR's of types RL and RR defined by the IGR's in Table 2.

The first step is to verify that the following is the list of all the sets of IRR outlines corresponding to (36)-(81). Note the multiple outlines corresponding to single IRR's that have multiple origins.

Table 5: Members of IRR(3) of types RL and RR and their corresponding sets of IRR outlines from Table 2

IRR(3) member	set of IRR outlines	multiplicity
(36)	$\{4, 8\}$	1
(37)	$\{1, 5\}$	1
(38)	$\{2, 6\}$	2
(39)	$\{3, 7\}$	1
(45)	$\{19, 21\}$	1
(46)	$\{10, 13\}$	1
(47)	$\{15, 17\}$	1
(57)	$\{55, 58\}$	1
(58)	$\{62, 67\}$	1
(59)	$\{37\}$	1
(60)	$\{51\}$	1
(61)	$\{54, 57\}$	1
(61)	$\{63, 68\}$	1
(62)	$\{34\}$	1
(63)	$\{49\}$	1
(71).1	$\{39\}$	1
(71).2	$\{38\}$	1
(72).1	$\{26\}$	1
(72).2	$\{23\}$	1
(73)	$\{61, 66\}$	1
(74)	$\{32\}$	1
(75)	$\{48\}$	1
(81).1	$\{44\}$	1
(81).2	$\{30\}$	1

For example the single IRR (36) corresponds to the set $\mathtt{S} = \{4, 8\}$, and it leads under $\mathtt{F}$ (see Table 2) to $\{19, 21\}$, $\{11, 14\}$ and $\{15, 17\}$ which each correspond to single IRR's of length 4 corresponding to $\alpha = \mathtt{a}, \mathtt{c}, \mathtt{d}$ respectively. Regardless of the length of the IRR corresponding to $\{4, 8\}$ a similar argument could be made giving the same IRR outlines. This is the importance of using IRR outlines. As another example, a single IRR of length 3 corresponds to $\{32\}$, which leads to $\{62\}$, $\{67\}$, $\{37\}$ and $\{51\}$. The first two of these have the same value of α ($\mathtt{a}$) so they represent the same IRR which is represented by $\{62, 67\}$. Thus $\{32\}$ leads under $\mathtt{F}$ to $\{62, 67\}, \{37\}$ and $\{51\}$. Likewise each IRR matching $\{32\}$ of length $\mathtt{n}$, under $\mathtt{F}$, will lead to 3 IRR's of length $\mathtt{n} + 1$ matching $\{62, 67\}, \{37\}$ and $\{51\}$. Collecting all these results, one application of $\mathtt{F}$ gives the following

Table 6

set of IRR outlines	set of sets of IRR outlines after $\mathtt{F}$	Set of LHS states

representing an IRR(3)		
{1, 5}	{19, 21}, {10, 13}, {15, 17}	1
{2, 6}	{20, 22}, {9, 12}, {16, 18}	1
{3, 7}	{19, 21}, {10, 13}, {15, 17}	1
{4, 8}	{19, 21}, {11, 14}, {15, 17}	1
{10, 13}	$\emptyset$	2
{15, 17}	{2, 6}	2
{19, 21}	{4, 8}	2
{23}	{59, 64}, {32}, {47}	4
{26}	$\emptyset$	4
{30}	{61, 66}, {32}, {48}	5
{32}	{62, 67}, {37}, {51}	4
{34}	{62, 67}, {36}, {50}	3
{37}	$\emptyset$	3
{38}	{40}, {24}	4
{39}	$\emptyset$	4
{44}	{39}, {26}	5
{48}	$\emptyset$	4
{49}	{42}, {28}	3
{51}	$\emptyset$	3
{54, 57}	$\emptyset$	3
{55, 58}	$\emptyset$	3
{61, 66}	$\emptyset$	4
{62, 67}	{55, 58}	3
{63, 68}	$\emptyset$	3

Another application of `F` to these results gives the following results:

Table 7

set of IRR outlines	set of sets of IRR outlines after `F`	Set of LHS states
{2, 6}	{20, 22}, {9, 12}, {16, 18}	2
{4, 8}	{19, 21}, {11, 14}, {15, 17}	2
{9, 12}	{1, 5}	1
{10, 13}	$\emptyset$	1
{11, 14}	$\emptyset$	1
{15, 17}	{2, 6}	1
{16, 18}	{2, 6}	1
{19, 21}	{4, 8}	1
{20, 22}	$\emptyset$	1
{24}	{60, 65}, {33}, {46}	4
{26}	$\emptyset$	5
{28}	{60, 65}, {33}, {46}	3
{32}	{62, 67}, {37}, {51}	{4, 5}
{36}	$\emptyset$	3

$\{37\}$	$\emptyset$	4
$\{39\}$	$\emptyset$	5
$\{40\}$	$\emptyset$	4
$\{42\}$	$\{41\}, \{25\}$	3
$\{47\}$	$\{43\}, \{27\}$	4
$\{48\}$	$\emptyset$	5
$\{50\}$	$\{43\}, \{27\}$	3
$\{51\}$	$\emptyset$	4
$\{55, 58\}$	$\emptyset$	3
$\{59, 64\}$	$\{53, 56\}$	4
$\{61, 66\}$	$\emptyset$	5
$\{62, 67\}$	$\{55, 58\}$	$\{3, 4\}$

In the following table are given all these results i.e. the sets of IRR outlines in Table 5 and those obtained by repeatedly applying `F` to these until closure i.e. any set of IRR outlines in column 2 are included in column 1. The set of LHS states for each IRR outline includes all that arise in this process.

Table 8: Relations between sets of IRR outlines under F

Original set of IRR outlines	Set of sets of IRR outlines derived by `F`	Set of LHS states
$\{1,5\}$	$\{19,21\},\{10,13\},\{15,17\}$	$\{1,2\}$
$\{2,6\}$	$\{20,22\},\{9,12\},\{16,18\}$	$\{1,2\}$
$\{3,7\}$	$\{19,21\},\{10,13\},\{15,17\}$	$\{1\}$
$\{4,8\}$	$\{19,21\},\{11,14\},\{15,17\}$	$\{1,2\}$
$\{9,12\}$	$\{1,5\}$	$\{1,2\}$
$\{10,13\}$	$\emptyset$	$\{1,2\}$
$\{11,14\}$	$\emptyset$	$\{1,2\}$
$\{15,17\}$	$\{2,6\}$	$\{1,2\}$
$\{16,18\}$	$\{2,6\}$	$\{1,2\}$
$\{19,21\}$	$\{4,8\}$	$\{1,2\}$
$\{20,22\}$	$\emptyset$	$\{1,2\}$
$\{23\}$	$\{59,64\},\{32\},\{47\}$	$\{4\}$
$\{24\}$	$\{60,65\},\{33\},\{46\}$	$\{4\}$
$\{25\}$	$\emptyset$	$\{3,4\}$
$\{26\}$	$\emptyset$	$\{3,4,5\}$
$\{27\}$	$\{61,66\},\{32\},\{48\}$	$\{3,4\}$
$\{28\}$	$\{60,65\},\{33\},\{46\}$	$\{3\}$
$\{29\}$	$\emptyset$	$\{3,4\}$
$\{30\}$	$\{61,66\},\{32\},\{48\}$	$\{5\}$
$\{31\}$	$\emptyset$	$\{4\}$
$\{32\}$	$\{62,67\},\{37\},\{51\}$	$\{3,4,5\}$
$\{33\}$	$\{62,67\},\{35\},\{52\}$	$\{3,4\}$
$\{34\}$	$\{62,67\},\{36\},\{50\}$	$\{3\}$
$\{35\}$	$\emptyset$	$\{3,4\}$
$\{36\}$	$\emptyset$	$\{3\}$
$\{37\}$	$\emptyset$	$\{3,4,5\}$
$\{38\}$	$\{40\},\{24\}$	$\{4\}$
$\{39\}$	$\emptyset$	$\{3,4,5\}$
$\{40\}$	$\emptyset$	$\{4\}$
$\{41\}$	$\emptyset$	$\{3,4\}$
$\{42\}$	$\{41\},\{25\}$	$\{3,4\}$
$\{43\}$	$\{39\},\{26\}$	$\{3,4\}$
$\{44\}$	$\{39\},\{26\}$	$\{5\}$
$\{45\}$	$\{40\},\{24\}$	$\{4\}$
$\{46\}$	$\{42\},\{29\}$	$\{3,4\}$
$\{47\}$	$\{43\},\{27\}$	$\{4\}$
$\{48\}$	$\emptyset$	$\{3,4,5\}$
$\{49\}$	$\{42\},\{28\},$	$\{3\}$
$\{50\}$	$\{43\},\{27\}$	$\{3\}$
$\{51\}$	$\emptyset$	$\{3,4,5\}$

$\{52\}$	$\emptyset$	$\{3,4\}$
$\{53,56\}$	$\{45\},\{31\}$	$\{4\}$
$\{54,57\}$	$\emptyset$	$\{3\}$
$\{55,58\}$	$\emptyset$	$\{3,4,5\}$
$\{59,64\}$	$\{53,56\}$	$\{4\}$
$\{60,65\}$	$\emptyset$	$\{3,4\}$
$\{61,66\}$	$\emptyset$	$\{3,4,5\}$
$\{62,67\}$	$\{55,58\}$	$\{3,4,5\}$
$\{63,68\}$	$\emptyset$	$\{3\}$

By drawing out the directed graph corresponding to Table 8 it is easy to show that it has 4 connected components (denoted by `C1-C4` respectively), the first 11 rows of the following Table 9, $\{54,57\}$,$\{63,68\}$, and the remaining nodes. The only non-trivial strongly connected component (SCC) `S1` has a subset of the nodes of `C1`. A strongly connected component (SCC) of a directed graph is a subset of the nodes of the directed graph and its associated edges such that every node of the SCC is connected to every other node of the SCC, directly or indirectly in both directions, and no other nodes can be added to this subset and retain this property, thus every node in an SCC is in a cycle i.e. a path back to itself. Single nodes can constitute an SCC with or without an edge connecting it to itself, and every node of a directed graph is in an SCC containing it. A non-trivial SCC (NSCC) is an SCC containing more than one node.

The single NSCC implicit in the directed graph defined by Table 8 has been verified by implementing Tarjan's algorithm [4] in the programming language D [5]. It was very satisfying to implement this algorithm because of its elegance (and ease of programming in D) and speed to do a task that is rather awkward by hand.

The importance of cycles is that they show that there are an infinite number of IRR's and allow a recursive definition to be made which defines an infinite subset of the IRR's. Table 9 lists (1) each set (node) in Table 8, (2) the SCC it is in except for the trivial cases (3) whether or not there is a path from the starting node to a node in `S1`, (4) "order" which represents the order of derivation. This is the length of the longest path from the starting node to any node in `S1` or a terminating node.

Table 9 could presumably be obtained by an extension of Tarjan's algorithm for obtaining the SCC's from any directed graph. The other columns of Table 9 can be verified easily in any order such that "order" is non-decreasing.

Table 9: Resolution of the directed graph defined by Table 8

set	SCC	$\to$ S1	order
{1, 5}	S1	✓	1
{2, 6}	S1	✓	1
{3, 7}	✗	✓	1
{4, 8}	S1	✓	1
{9, 12}	S1	✓	1
{10, 13}	✗	✗	0
{11, 14}	✗	✗	0
{15, 17}	S1	✓	1
{16, 18}	S1	✓	1
{19, 21}	S1	✓	1
{20, 22}	✗	✗	0
{23}	✗	✗	7
{24}	✗	✗	3
{25}	✗	✗	0
{26}	✗	✗	0
{27}	✗	✗	3
{28}	✗	✗	3
{29}	✗	✗	0
{30}	✗	✗	3
{31}	✗	✗	0
{32}	✗	✗	2
{33}	✗	✗	2
{34}	✗	✗	5
{35}	✗	✗	0
{36}	✗	✗	0
{37}	✗	✗	0
{38}	✗	✗	4
{39}	✗	✗	0
{40}	✗	✗	0
{41}	✗	✗	0
{42}	✗	✗	1
{43}	✗	✗	1
{44}	✗	✗	1
{45}	✗	✗	4
{46}	✗	✗	2
{47}	✗	✗	4
{48}	✗	✗	0
{49}	✗	✗	4
{50}	✗	✗	4
{51}	✗	✗	0
{52}	✗	✗	0

{53, 56}	✗	✗	5
{54, 57}	✗	✗	0
{55, 58}	✗	✗	0
{59, 64}	✗	✗	6
{60, 65}	✗	✗	0
{61, 66}	✗	✗	0
{62, 67}	✗	✗	1
{63, 68}	✗	✗	0

This in particular characterises the cycles that lead to infinite numbers of IRR's. Now the numbers of IRR(n) for a range of values of $n \geq 3$ can be found starting from each IRR outline in Table 5 by repeatedly referring to Table 8 and totalling the results. Table 10 following has the results starting from (36).

Table 10: The number of IRR's of length n derived by repeated applications of IGR's in Table 2 starting from a single IRR matching {4, 8} of length 3.

IRR				n				
outline	3	4	5	6	7	8	9	10
{1, 5}	0	0	0	0	1	0	2	0
{2, 6}	0	0	1	0	2	0	4	0
{3, 7}	0	0	0	0	0	0	0	0
{4, 8}	1	0	1	0	1	0	2	0
{9, 12}	0	0	0	1	0	2	0	4
{10, 13}	0	0	0	0	0	1	0	2
{11, 14}	0	1	0	1	0	1	0	2
{15, 17}	0	1	0	1	0	2	0	4
{16, 18}	0	0	0	1	0	2	0	4
{19, 21}	0	1	0	1	0	2	0	4
{20, 22}	0	0	0	1	0	2	0	4
Total	1	3	2	6	4	12	8	24

Similar calculations were done with the other IRR(3) starting points and totalled to give the following.

Table 11: The number of IRR's of length n derived by repeated applications of IGR's in Table 2 starting from all the IRR(3).

Starting				n				
IRR outline	3	4	5	6	7	8	9	10
{4, 8}	1	3	2	6	4	12	8	24
{1, 5}	1	3	2	6	4	12	8	24
{2, 6} × 2	2	6	4	12	8	24	16	48
{3, 7}	1	3	2	6	4	12	8	24
{19, 21}	1	1	3	2	6	4	12	8

$\{\mathtt{15,17}\}$	1	1	3	2	6	4	12	8
order $0 \times \mathtt{10}$	10	0	0	0	0	0	0	0
$\{\mathtt{62,67}\}$	1	1	0	0	0	0	0	0
$\{\mathtt{44}\}$	1	2	0	0	0	0	0	0
$\{\mathtt{32}\}$	1	3	1	0	0	0	0	0
$\{\mathtt{30}\}$	1	3	3	1	0	0	0	0
$\{\mathtt{49}\}$	1	2	5	5	3	0	0	0
$\{\mathtt{38}\}$	1	2	3	5	3	0	0	0
$\{\mathtt{34}\}$	1	3	3	5	3	1	0	0
$\{\mathtt{23}\}$	1	3	6	8	5	4	5	3
Total	25	36	37	58	46	73	69	139

The "order" 0 is the set of 10 IRR(3) that gave no IRR with larger $\mathtt{n}$ in Table 6. The remaining results from $\{\mathtt{62,67}\}$ onwards were done in increasing order of "order". As an example starting from a single IRR in $\{\mathtt{30}\}$ of length 3 gives under $\mathtt{F}$, 3 members of IRR(4) one in each of $\{\mathtt{61,66}\}, \{\mathtt{32}\}, \{\mathtt{48}\}$. Of these only $\{\mathtt{32}\}$ gives further IRR's which are 3 members of IRR(5) one in each of $\{\mathtt{62,67}\}, \{\mathtt{37}\}, \{\mathtt{51}\}$, and of these only the first entails a member of IRR(6) in $\{\mathtt{55,58}\}$ which ends the derivation sequence. Thus the counts for IRR are 1,3,3,1, then zeros for $\mathtt{n} = 3,4,5,6,\ldots$ respectively.

Similar calculations can be done for the mirror image results i.e. those with the $\ldots$ on the left (Table 22). Counting the IRR obtained from Table 22 could of course be done likewise. Table 12 contains the RHS's listed as `RHS` in Tables 2 and 22.

Table 12: RHS's for the non-IRR($\mathtt{n}+\mathtt{1}$) triplets in Tables 2 and 22 i.e. where the new origin does not justify reachability of the LHS.

RHS of IRR(n)	symbol α				
	a	b	c	d	e
2_ab...	3dbc...	4_cab...	3abc...	3cbc...	3_cab...
2_ae...	5_ccc...	4_cae...	3_bcc...	1abd...	3_cae...
2_ec...	1cbd...	4_cec...	1dbd...	1abd...	3_cec...
3_bc...	3cbc...	4_cbc...	2_abc...	5_cbc...	2bab...
4_ca...	3_bca...	4bcc...	1abc...	5_cca...	4aac...
4_cb...	3_bcb...	2bab...	3abc...	5_ccb...	3cbc...
4_ce...	3_bce...	3bcb...	3_bcc...	5_cce...	3aab...
5_cc...	2_ecc...	3_ecc...	1dbd...	4_acc	1dbd...
5_ce...	2_ece...	3_ece...	5_ccc...	4_ace...	5_ccc...
1...ba_	2...bcb_	4...cbd	4...cab	2...bab_	2...bab_
1...bc_	2...bdb_	2...bdb_	4...caa	2...bcb_	2...bcb_
1...bd_	2...bab_	3...ecd	3...eca	2...bdb_	2...bdb_
2...ab_	1...abc_	3...abc_	1...abd_	1...aba_	3...bcc
2...cb_	1...cbc_	3...cbc_	1...cbd_	1...cba_	1...abd_
2...db_	1...dbc_	3...dbc_	1...dbd_	1...dba_	5...ccc
3...aa_	4...aac_	3...cbc_	2...adb_	1...cbd_	3...aab_
3...bc_	4...bcc_	2...bab_	2...bab_	2...bdb_	3...bcb_
4...ac_	3...dbc_	4...acb_	3...acc_	2...adb_	5...aca_
4...cb_	3...abc_	4...cbb_	3...cbc_	2...aec	5...cba_
4...cc_	3...abc_	4...ccb_	3...ccc_	2...cdb_	5...cca_

9.1 Abbreviating the tables of IRR-generating rules

Tables 2 and 22 have a lot of structure i.e. regularities that result directly from how they were obtained. In what follows use will be made of this, so I here explain this by means of a theorem with its proof.

Theorem 9.1. *In Tables 2 and 22 the rows can be put into groups such that the set of abbreviations of the origins of the IRR(n+1) for each member of the group are the same, and in this set, the symbol adjacent to the one with the pointer is the same (*`sy2`*) and the set of abbreviations of the origins of the IRR(n) for the group all have (1) the same state and (2) the same symbol at the pointer which is also* `sy2`*.*

Proof. In this proof, `st` and `sy` with subscripts will stand for a state and a symbol respectively. As usual, α represents an arbitrary symbol that plays the role of α in the proof of Theorem 2.2. The proof as written is for the case where the arbitrary symbols ... are on the right (Table 2). It can be adapted to the other case by reversing the strings of symbols everywhere.

Consider the set of abbreviated origins $\mathtt{S}$ of IRR($\mathtt{n}$) that lead to the abbreviated origin $\mathtt{st_1\underline{sy_1}sy_2}\ldots$ of IRR($\mathtt{n}+\mathtt{1}$) in Table 2 for a particular value of α. They are obtained by a single forward TM step in each case (but complicated by the abbreviation). Therefore members of $\mathtt{S}$ can be obtained from $\mathtt{st_1\underline{sy_1}sy_2}\ldots$ using (1), so $\mathtt{sy_2}$ is unaffected and must appear in each member of $\mathtt{S}$ at the pointer because the TM is going right in this case. Thus forward computation by 1 step from $\mathtt{st_1\underline{sy_1}sy_2}\ldots$ must match each member of $\mathtt{S}$ and in general $\mathtt{S}$ is in $\{\mathtt{st_2\underline{sy_2}sy_{3i}}\ldots\}_{\mathtt{i}=1}^{\mathtt{k}}$ for some integer $\mathtt{k}$ where $\mathtt{i}$ indexes the groups mentioned above, and $\mathtt{st_1\underline{sy_1}}$ leads under 1 TM step to state $\mathtt{st_2}$. Now we need to find all the origins of IRR($\mathtt{n}+\mathtt{1}$), i.e. what leads to $\mathtt{st_2}\alpha\mathtt{\underline{sy_2}sy_{3i}}\ldots$ which is $\mathtt{x} = \mathtt{st_3\underline{sy_4}sy_2sy_{3i}}\ldots$ where say $\mathtt{st_2}\alpha_- \leftarrow \mathtt{st_3\underline{sy_4}}$. For each such pair $(\mathtt{st_3}, \mathtt{sy_4})$ the result is shortened to length 2 as $\mathtt{st_3\underline{sy_4}sy_2}\ldots$, which is independent of $\mathtt{i}$. □

For example $\mathtt{S} = \{\mathtt{1\underline{d}a}\ldots, \mathtt{1\underline{d}c}\ldots, \mathtt{1\underline{d}d}\ldots\}$ as abbreviated origins of IRR($\mathtt{n}$) give rise to the abbreviated origin $\mathtt{2\underline{d}d}\ldots$ in IRR($\mathtt{n}+\mathtt{1}$) for $\alpha = \mathtt{a}$, (which arises from $\mathtt{1}\alpha\mathtt{\underline{d}a}\ldots \leftarrow \mathtt{2\underline{d}da}\ldots$, $\mathtt{1}\alpha\mathtt{\underline{d}c} \leftarrow \mathtt{2\underline{d}dc}\ldots$ and $\mathtt{1}\alpha\mathtt{\underline{d}d}\ldots \leftarrow \mathtt{2\underline{d}dd}\ldots$ for $\alpha = \mathtt{a}$ which are each shortened to $\mathtt{2\underline{d}d}\ldots$ in Table 2), then with $\alpha = \mathtt{c}$ likewise with the same set $\mathtt{S}$ of abbreviated origins for IRR($\mathtt{n}$) we get $\mathtt{1}\alpha\mathtt{\underline{d}a}\ldots \leftarrow \mathtt{2\underline{a}da}\ldots, \mathtt{1}\alpha\mathtt{\underline{d}c} \leftarrow \mathtt{2\underline{a}dc}\ldots, \mathtt{1}\alpha\mathtt{\underline{d}d}\ldots \leftarrow \mathtt{2\underline{a}dd}\ldots$ which are all shortened to $\mathtt{2\underline{a}d}\ldots$ as the corresponding abbreviated origin in IRR($\mathtt{n}+\mathtt{1}$) etc.. This fact allows the table of the relationships between the derived origin (in IRR($\mathtt{n}+\mathtt{1}$)) and the original origins (in IRR($\mathtt{n}$)) to be written succinctly as follows, where the incidence matrix has a 1 indicating each possible combination of RHS of IRR($\mathtt{n}$) and β, and a zero where this combination does not occur.

Table 13: Origins of IRR($\mathtt{n}+\mathtt{1}$) derived from origins of IRR($\mathtt{n}$) and α, and exceptions for β

Origins and RHS's of IRR($\mathtt{n}$)	β and incidence matrix	Origins of IRR($\mathtt{n}+\mathtt{1}$) for symbol α a	b	c	d	e	β
$\mathtt{1\underline{d}}\beta\ldots$	a c d	$\mathtt{2\underline{d}d}\ldots$		$\mathtt{2\underline{a}d}\ldots$	$\mathtt{2\underline{c}d}\ldots$		
3_ec...	0 1 0						
4_ca...	1 0 1						
4_cb...	0 0 1						
$\mathtt{1\underline{e}}\beta\ldots$	a c d	$\mathtt{2\underline{d}e}\ldots$		$\mathtt{2\underline{a}e}\ldots$	$\mathtt{2\underline{c}e}\ldots$		
3_ec...	0 1 0						
4_ca...	1 0 1						
4_cb...	0 0 1						
$\mathtt{2\underline{a}}\beta\ldots$	d e		$\mathtt{1\underline{d}a}\ldots$				de
2_ae...	1 1		$\mathtt{1\underline{e}a}\ldots$				
$\mathtt{2\underline{c}}\beta\ldots$	d e		$\mathtt{1\underline{d}c}\ldots$				de
5_cc...	1 1		$\mathtt{1\underline{e}c}\ldots$				
5_ce...	1 1						
$\mathtt{2\underline{d}}\beta\ldots$	d e		$\mathtt{1\underline{d}d}\ldots$				de
3_bc...	1 1		$\mathtt{1\underline{e}d}\ldots$				
$\mathtt{3\underline{a}}\beta\ldots$	b c e	$\mathtt{5\underline{c}a}\ldots$	$\mathtt{3\underline{e}a}\ldots$	$\mathtt{4\underline{c}a}\ldots$			bce
2_ab...	0 1 1	$\mathtt{5\underline{e}a}\ldots$					
2_ae...	1 0 0						
3_bc...	1 1 0						

$3\underline{e}\beta\ldots$ 4_ca... 4_cb... 4_ce...	a e 1 0 1 0 0 1	$5\underline{c}e\ldots$ $5\underline{e}e\ldots$	$3\underline{e}e\ldots$	$4\underline{c}e\ldots$			ae
$4\underline{b}\beta\ldots$ 4_ca... 4_cb... 4_ce...	b c e 0 1 0 0 1 1 1 0 1		$4\underline{b}b\ldots$	$3\underline{a}b\ldots$			c
$4\underline{c}\beta\ldots$ 2_ab... 2_ae... 3_bc...	a e 1 0 0 1 1 1		$4\underline{b}c\ldots$	$3\underline{a}c\ldots$			a
$4\underline{e}\beta\ldots$ 2_ec...	c e 1 1		$4\underline{b}e\ldots$	$3\underline{a}e\ldots$			c
$5\underline{c}\beta\ldots$ 3_bc... 5_cc...	a e 0 1 1 0	$4\underline{e}c\ldots$					
$5\underline{e}\beta\ldots$ 3_bc 5_cc	a e 0 1 1 0	$4\underline{e}e\ldots$					
$1\ldots\beta\underline{a}$ 1...ba_ 1...bd_ 2...db_	b e 1 1 1 0 1 1						
$1\ldots\beta\underline{b}$ 1...ba_ 1...bd_ 2...cb_ 2...db_	a d 1 1 0 1 0 1 1 1						d
$1\ldots\beta\underline{c}$ 1...bc_ 2...ab_ 2...db_ 3...bc_ 4...ac_ 4...cc_	b c d 1 1 1 1 0 1 0 1 1 0 1 1 1 0 1 1 0 1						cd
$2\ldots\beta\underline{b}$ 1...bd_ 2...ab_ 2...db_ 3...bc_ 3...cc_	a d 1 1 1 1 0 1 1 0 0 1	$3\ldots b\underline{c}$			$1\ldots b\underline{a}$	$5\ldots b\underline{a}$	
$2\ldots\beta\underline{e}$ 1...bd_ 2...ab_ 2...db_ 3...cc_	b c d 1 1 1 1 0 1 1 0 1 0 1 1	$3\ldots e\underline{c}$			$1\ldots e\underline{a}$	$5\ldots e\underline{a}$	b
$3\ldots\beta\underline{b}$ 1...bd_ 2...ab_ 2...db_ 3...bc_ 3...cc_	a d 1 1 1 1 0 1 1 0 0 1	$1\ldots b\underline{c}$	$4\ldots b\underline{a}$	$2\ldots b\underline{e}$		$5\ldots b\underline{b}$	a
$3\ldots\beta\underline{c}$ 1...bc_ 2...ab_ 4...ac_ 4...cc_	b e 1 1 1 1 1 0 1 1	$1\ldots c\underline{c}$	$4\ldots c\underline{a}$	$2\ldots c\underline{e}$		$5\ldots c\underline{b}$	b

3...β$\underline{\mathtt{d}}$	a b d		
1...bc_	1 1 0		
1...bd_	1 1 1		
2...ab_	0 0 1	1...d$\underline{\mathtt{c}}$ 4...d$\underline{\mathtt{a}}$ 2...d$\underline{\mathtt{e}}$ 5...d$\underline{\mathtt{b}}$	ab
3...aa_	0 1 0		
3...cc_	0 0 1		
4...cc_	0 1 0		
4...β$\underline{\mathtt{a}}$	b c d		
2...ab_	1 0 1		
2...db_	0 1 1	5...a$\underline{\mathtt{d}}$ 2...a$\underline{\mathtt{b}}$ 3...a$\underline{\mathtt{b}}$ 1...a$\underline{\mathtt{b}}$	bc
3...bc_	1 1 1		
4...cb_	0 1 1		
4...β$\underline{\mathtt{d}}$	a b d		
1...bc_	1 1 0		
1...bd_	1 1 1		
2...ab_	0 0 1	5...d$\underline{\mathtt{d}}$ 2...d$\underline{\mathtt{b}}$ 3...d$\underline{\mathtt{b}}$ 1...d$\underline{\mathtt{b}}$	b
3...aa_	0 1 0		
3...cc_	0 0 1		
4...cc_	0 1 0		
5...β$\underline{\mathtt{a}}$	b e		
2...db_	1 1	3...a$\underline{\mathtt{d}}$ 4...a$\underline{\mathtt{d}}$	
3...ab_	1 0		
3...cb_	1 1		
5...β$\underline{\mathtt{b}}$	b c d		
1...bd_	0 1 0		
2...cb_	0 1 1		
2...db_	1 0 1	3...b$\underline{\mathtt{d}}$ 4...b$\underline{\mathtt{d}}$	
3...ab_	1 0 1		
3...cb_	1 0 1		
5...ca_	0 1 1		
5...β$\underline{\mathtt{d}}$	a d		
1...bc_	1 1		
2...ab_	0 1		
2...db_	0 1	3...d$\underline{\mathtt{d}}$ 4...d$\underline{\mathtt{d}}$	a
3...bc_	1 1		
4...ac_	0 1		
4...cc_	1 1		

Table 13 together with Table 14 for the RHS's, being an abbreviation of tables 2 and 22, is consistent with the closure property i.e. every abbreviated origin and RHS an IRR appearing on the right also appears on the left. The point of this is to ensure that every IRR that can be obtained from any member of IRR(3) by repeated applications of IGR's of length 2 are represented, together with the IGR's to generate them, in Tables 13 and 14. For example 2$\underline{\mathtt{d}}$d... on the right in the first major row Table 13 with $\alpha = \mathtt{a}$ also appears on the left in the fifth major row with $\beta = \mathtt{d}$. Also the RHS's associated with 2$\underline{\mathtt{d}}$d... and $\alpha = \mathtt{a}$ which are just 3_bc as Table 14 shows after applying $\alpha = \mathtt{a}$ to 3_ec..., 4_ca... and 4_cb... which are the RHS's in the first major row of Table 13, (ignoring the 3ca... because it leads to an IRR of non-extendable type RR) is also the RHS of IRR(n) in the fifth major row of Table 13.

Table 14: RHS's of IRR($\mathtt{n}+\mathtt{1}$) derived from RHS's of IRR($\mathtt{n}$) and α

	RHS's of IRR($\mathtt{n}+\mathtt{1}$) for symbol α				
RHS's of IRR($\mathtt{n}$)	a	b	c	d	e
2_ab...	3db...	4_ca...	3ab...		
2_ae...	5_cc...	4_ca...	3_bc...		
2_ec...	1cb...	4_ce...	1db...		
3_bc...	3cb...	4_cb...	2_ab...		
3_cb...	2ca...				
3_ec...	3ca...		2_ae...	5_ce...	
4_ca...	3_bc...	4bc...	1ab...	5_cc...	
4_cb...	3_bc...	2ba...	3ab...	5_cc...	
4_ce...	3_bc...	3bc...	3_bc...		
5_cc...	2_ec...	3_ec...			
5_ce...		3_ec...			
1...bc_	2...db_	2...db_	4...aa	2...cb_	2...cb_
1...bd_	2...ab_	3...cd	3...ca	2...db_	2...db_
2...ab_	1...bc_	3...bc_	1...bd_	1...ba_	3...cc
2...cb_	1...bc_		1...bd_	1...ba_	
2...db_	1...bc_	3...bc_	1...bd_	1...ba_	5...cc
3...aa_	4...ac_	3...bc_	2...db_	1...bd_	3...ab_
3...ab_	4...bc_		3...ca		3...bb_
3...cb_			1...bc_		
3...bc_	4...cc_	2...ab_	2...ab_	2...db_	3...cb_
3...cc_	4...cc_	2...ab_	2...ab_	2...db_	3...cb_
4...cb_	3...bc_		3...bc_	2...ec	
4...ac_	3...bc_	4...cb_	3...cc_	2...db_	5...ca_
4...cc_	3...bc_	4...cb_	3...cc_	2...db_	5...ca_
5...ca_			3...aa_		

Table 15: Origins of IRR($\mathtt{n}+\mathtt{1}$) derived from origins of IRR($\mathtt{n}$) for $\mathtt{k} = \mathtt{2}$ exceptions

Origin of IRR($\mathtt{n}$)	Origins of IRR($\mathtt{n}+\mathtt{1}$)
$\mathtt{2}\alpha\underline{\mathtt{a}}\mathtt{d}\ldots$	$\mathtt{1}\alpha\mathtt{a}\underline{\mathtt{a}}\ldots$
$\mathtt{2}\alpha\underline{\mathtt{a}}\mathtt{e}\ldots$	$\mathtt{5}\alpha\mathtt{a}\underline{\mathtt{a}}\ldots$
$\mathtt{2}\alpha\underline{\mathtt{c}}\mathtt{d}\ldots$	$\mathtt{1}\alpha\mathtt{c}\underline{\mathtt{a}}\ldots$
$\mathtt{2}\alpha\underline{\mathtt{c}}\mathtt{e}\ldots$	$\mathtt{5}\alpha\mathtt{c}\underline{\mathtt{a}}\ldots$
$\mathtt{2}\alpha\underline{\mathtt{d}}\mathtt{d}\ldots$	$\mathtt{1}\alpha\mathtt{d}\underline{\mathtt{a}}\ldots$
$\mathtt{2}\alpha\underline{\mathtt{d}}\mathtt{e}\ldots$	$\mathtt{5}\alpha\mathtt{d}\underline{\mathtt{a}}\ldots$
$\mathtt{3}\alpha\underline{\mathtt{a}}\mathtt{b}\ldots$	$\mathtt{4}\alpha\mathtt{a}\underline{\mathtt{a}}\ldots$
$\mathtt{3}\alpha\underline{\mathtt{a}}\mathtt{c}\ldots$	$\mathtt{2}\alpha\mathtt{a}\underline{\mathtt{e}}\ldots$
$\mathtt{3}\alpha\underline{\mathtt{a}}\mathtt{e}\ldots$	$\mathtt{5}\alpha\mathtt{a}\underline{\mathtt{b}}\ldots$
$\mathtt{3}\alpha\underline{\mathtt{e}}\mathtt{a}\ldots$	$\mathtt{1}\alpha\mathtt{e}\underline{\mathtt{c}}\ldots$
$\mathtt{3}\alpha\underline{\mathtt{e}}\mathtt{e}\ldots$	$\mathtt{5}\alpha\mathtt{e}\underline{\mathtt{b}}\ldots$
$\mathtt{4}\alpha\underline{\mathtt{b}}\mathtt{c}\ldots$	$\mathtt{2}\alpha\mathtt{b}\underline{\mathtt{b}}\ldots, \mathtt{3}\alpha\mathtt{b}\underline{\mathtt{b}}\ldots$
$\mathtt{4}\alpha\underline{\mathtt{c}}\mathtt{a}\ldots$	$\mathtt{5}\alpha\mathtt{c}\underline{\mathtt{d}}\ldots$
$\mathtt{4}\alpha\underline{\mathtt{e}}\mathtt{c}\ldots$	$\mathtt{2}\alpha\mathtt{e}\underline{\mathtt{b}}\ldots, \mathtt{3}\alpha\mathtt{e}\underline{\mathtt{b}}\ldots$
$\mathtt{1}\ldots\mathtt{d}\underline{\mathtt{b}}\alpha$	$\mathtt{2}\ldots\underline{\mathtt{c}}\mathtt{b}\alpha$
$\mathtt{1}\ldots\mathtt{c}\underline{\mathtt{c}}\alpha$	$\mathtt{2}\ldots\underline{\mathtt{a}}\mathtt{c}\alpha$
$\mathtt{1}\ldots\mathtt{d}\underline{\mathtt{c}}\alpha$	$\mathtt{2}\ldots\underline{\mathtt{c}}\mathtt{c}\alpha$
$\mathtt{2}\ldots\mathtt{b}\underline{\mathtt{e}}\alpha$	$\mathtt{1}\ldots\underline{\mathtt{d}}\mathtt{e}\alpha, \mathtt{1}\ldots\underline{\mathtt{e}}\mathtt{e}\alpha$
$\mathtt{3}\ldots\mathtt{a}\underline{\mathtt{b}}\alpha$	$\mathtt{5}\ldots\underline{\mathtt{c}}\mathtt{b}\alpha, \mathtt{5}\ldots\underline{\mathtt{e}}\mathtt{b}\alpha$
$\mathtt{3}\ldots\mathtt{b}\underline{\mathtt{c}}\alpha$	$\mathtt{3}\ldots\underline{\mathtt{e}}\mathtt{c}\alpha$
$\mathtt{3}\ldots\mathtt{a}\underline{\mathtt{d}}\alpha$	$\mathtt{5}\ldots\underline{\mathtt{c}}\mathtt{d}\alpha, \mathtt{5}\ldots\underline{\mathtt{e}}\mathtt{d}\alpha$
$\mathtt{3}\ldots\mathtt{b}\underline{\mathtt{d}}\alpha$	$\mathtt{3}\ldots\underline{\mathtt{e}}\mathtt{d}\alpha$
$\mathtt{4}\ldots\mathtt{b}\underline{\mathtt{a}}\alpha$	$\mathtt{4}\ldots\underline{\mathtt{b}}\mathtt{a}\alpha$
$\mathtt{4}\ldots\mathtt{c}\underline{\mathtt{a}}\alpha$	$\mathtt{3}\ldots\underline{\mathtt{a}}\mathtt{a}\alpha$

$\mathtt{4}\ldots\mathtt{b}\underline{\mathtt{d}}\alpha$	$\mathtt{4}\ldots\underline{\mathtt{b}}\mathtt{d}\alpha$
$\mathtt{5}\ldots\mathtt{a}\underline{\mathtt{d}}\alpha$	$\mathtt{4}\ldots\underline{\mathtt{e}}\mathtt{d}\alpha$

10 Dealing with the exceptions

In what follows, $\mathtt{IRR}(\mathtt{n},\mathtt{k})$ is the set of all distinct abbreviations of members of IRR($\mathtt{n}$) with $\mathtt{k} < \mathtt{n}$ symbols specified. The abbreviation is just truncation to leave the symbols closest to the pointer in the origin and showing the symbols in the same places in the RHS. The remaining deleted symbols are replaced by $\ldots$.

In order to overcome the restriction to $\mathtt{k} = \mathtt{2}$ symbols in the definition of Tables 13 and 14, a generalisation $\mathtt{t}(\mathtt{k})$ to any number $\mathtt{k}$ of specified symbols will be defined as follows. A member of IRR($\mathtt{n},\mathtt{k}$) in $\mathtt{t}(\mathtt{k})$ (first major column) is either a member of IRR($\mathtt{k}+\mathtt{1},\mathtt{k}$) or has been obtained by repeated applications of $\mathtt{F}$ starting from member of IRR($\mathtt{k}+\mathtt{1},\mathtt{k}$) using one or more IGR's of length $\mathtt{k}$ and truncating the result by one symbol for each IGR applied in order that the length of the strings of specified symbols remains equal to $\mathtt{k}$. Each abbreviated non-IRR($\mathtt{n}+\mathtt{1}$) rule $\mathtt{Y}$ in the rightmost major column in $\mathtt{t}(\mathtt{k})$ i.e. with the pointer at the opposite end of the string from the α (an exception) is obtained from a corresponding member of IRR($\mathtt{n},\mathtt{k}$) $\mathtt{X}$. $\mathtt{t}(\mathtt{k})$ is defined to have this property and is therefore obtained by a closure procedure which requires that every member of IRR($\mathtt{n}+\mathtt{1},\mathtt{k}$) in the rightmost major column also appears in the IRR($\mathtt{n}$) column. In either case an extra symbol can be added to $\mathtt{X}$ giving the full member of IRR($\mathtt{k}+\mathtt{1}$), or by not abbreviating by removing one symbol in the last step of the closure procedure. Then the computation of the new origin can be continued.

In Table 16, each origin of a member of IRR($\mathtt{n}+\mathtt{1}$) is an origin of a member of IRR($\mathtt{n}+\mathtt{1}$) in Table 13 and is therefore an origin of a member of IRR($\mathtt{n}$) in Table 13. For example every CS in $\mathtt{1}\underline{\mathtt{d}}\mathtt{da}\ldots$ in the first row of Table 16 is also in $\mathtt{1}\underline{\mathtt{d}}\mathtt{d}\ldots$ (the third major row of Table 13). This implies that any IRR deduced using the IGR's in Table 16 can be used by an IGR in Table 13 to deduce another IRR. This needs to be generalised.

This procedure is complicated by the abbreviation of Table 2 for $\mathtt{k} = \mathtt{2}$ as Table 13 with RHS's given in Table 14 and the new origins for the exceptions given in Table 15, and corresponding abbreviations for all the $\mathtt{t}(\mathtt{k})$ for $\mathtt{k} > \mathtt{2}$. Table 13 is a data structure that seems to be minimal i.e. without repetition, and Tables13, 14 and 15, and their extensions to higher values of $\mathtt{k}$ should be used in an automated procedure for doing this. This explains the complexity of the method, which is described next using an example.

Continuing from the exceptions in Table 13 to implement the next stage in applying Theorem 2.2 to the TM under study, consider the first IRR abbreviation with an exception in Table 13 i.e. $\mathtt{A} = \mathtt{2}\underline{\mathtt{a}}\mathtt{d}\ldots \rightarrow\rightarrow \mathtt{2}\underline{\ }\mathtt{ae}\ldots$. In order to do this one more symbol on the right is needed. Going back to see the context in which this could arise, Table 13 shows that it can arise via $\mathtt{F}$ from $\mathtt{1\underline{d}}\beta\ldots \twoheadrightarrow \mathtt{RHS}$ with $\alpha = \mathtt{c}$, where $\mathtt{RHS} \in \{\mathtt{3\_ec}\ldots, \mathtt{4\_ca}\ldots, \mathtt{4\_cb}\ldots\}$ and $\beta \in \{\mathtt{a}, \mathtt{c}, \mathtt{d}\}$, and 4 combinations of these are possible given by the incidence matrix. Then from Table 14 with $\alpha = \mathtt{c}$ and the RHS of IRR($\mathtt{n}+\mathtt{1}$) is $\mathtt{2\_ae}\ldots$ which shows that the RHS of IRR($\mathtt{n}$) is $\mathtt{3\_ec}\ldots$ and finally from Table 13 $\beta = \mathtt{c}$, showing that $\mathtt{1\underline{d}c}\ldots \twoheadrightarrow \mathtt{3\_ec}\ldots$ leads to $\mathtt{A}$. Alternatively it is easy to show that, in the notation of (91) with one extra symbol in the abbreviated IRR on the right,

$$\mathtt{1\underline{d}}\ldots \twoheadrightarrow \mathtt{3\_e}\ldots \overset{\mathtt{c}}{\Rightarrow} \mathtt{2\underline{a}d}\ldots \twoheadrightarrow \mathtt{2\_ae}\ldots \tag{608}$$

where the $\ldots$ represents the same string on the left of the $\twoheadrightarrow$ on both sides and similarly on the right. Because every IRR has a unique image under $\mathtt{F}^{-1}$ (Theorem 2.2) it follows that this can be obtained in this case by using (608) i.e. the uniqueness of $\mathtt{F}^{-1}$ for IRR's implies the uniqueness of $\mathtt{F}^{-1}$ for the abbreviated IRR $\mathtt{2\underline{a}d}\ldots \twoheadrightarrow \mathtt{2\_ae}\ldots$ And because $\mathtt{1\underline{d}}\ldots \twoheadrightarrow \mathtt{3\_e}\ldots$ is only represented in Table 13 by its subset $\mathtt{1\underline{d}c}\ldots \twoheadrightarrow \mathtt{3\_ec}\ldots$, this then is the only possible source of $\mathtt{A}$ via $\mathtt{F}$ in Table 13.

Now without deleting the extra symbol this derivation starts by applying the backward search algorithm as follows $\mathtt{1c\underline{d}c}\ldots \leftarrow \mathtt{2\underline{a}dc}\ldots$ and forward computation $\mathtt{3\underline{c}ec}\ldots \rightarrow \mathtt{2\_aec}\ldots$ thus the extended version of $\mathtt{A}$ is $\mathtt{2\underline{a}dc}\ldots \twoheadrightarrow \mathtt{2\_eac}\ldots$. Now returning to the original problem, applying $\mathtt{F}$ to this starting with the backward searching algorithm gives

$$\mathtt{2}\alpha\mathtt{\underline{a}dc}\ldots \left\{\begin{array}{l} \overset{\alpha=\mathtt{b}}{\leftarrow} \left\{\begin{array}{l} \mathtt{1\underline{d}adc}\ldots \\ \mathtt{1\underline{e}adc}\ldots \end{array}\right. \\ \\ \leftarrow \mathtt{1}\alpha\mathtt{a\underline{a}c}\ldots \leftarrow \mathtt{2}\alpha\mathtt{\underline{d}ac}\ldots \left\{\begin{array}{l} \overset{\alpha=\mathtt{b}}{\leftarrow} \left\{\begin{array}{l} \mathtt{1\underline{d}dac}\ldots \\ \mathtt{1\underline{e}dac}\ldots \end{array}\right. \\ \leftarrow \mathtt{3}\alpha\mathtt{d\underline{c}c}\ldots \leftarrow \mathtt{2}\alpha\mathtt{dc\underline{e}}\ldots \end{array}\right. \end{array}\right. \tag{609}$$

The first two results for $\alpha = \mathtt{b}$ are nothing new and can be found in Table 13, bearing in mind that the $\mathtt{c}$ is not involved so can be omitted. The result $\mathtt{2}\alpha\mathtt{\underline{a}dc}\ldots \leftarrow \mathtt{1}\alpha\mathtt{a\underline{a}c}\ldots$ found in Table 15 can now be extended as shown, and the first two of these results are in the first row of Table 16 after truncation to $\mathtt{k} = \mathtt{3}$ symbols. Combining this with the result on the third row of Table 17 with $\alpha = \mathtt{b}$ shows that the derived abbreviated IRR's are $\mathtt{1\underline{d}da}\ldots \twoheadrightarrow \mathtt{4\_cae}\ldots$ and $\mathtt{1\underline{e}da}\ldots \twoheadrightarrow \mathtt{4\_cae}\ldots$ Again there is an exception i.e. $\mathtt{2}\alpha\mathtt{\underline{a}dc}\ldots \leftarrow \mathtt{2}\alpha\mathtt{dc\underline{e}}\ldots$ which appears in the second row of Table 18.

The general procedure is, starting from $\mathtt{A} = \mathtt{O1} \twoheadrightarrow \mathtt{R1} \in \mathtt{IRR(n,k)}$ find $\mathtt{O2} \twoheadrightarrow \mathtt{R2} \in \mathtt{IRR(n,k)}$ which gives rise to it in $\mathtt{t(k)}$, its extension $\mathtt{A}' = \mathtt{O3} \twoheadrightarrow \mathtt{R3} \in \mathtt{IRR(n,k+1)}$ and finally the application of $\mathtt{F}$ to $\mathtt{A}'$ as follows:

1. Look up $\mathtt{O1}$ in the second major column of $\mathtt{t(k)}$ to find α, the new origin $\mathtt{O2}$ in the left major column of $\mathtt{t(k)}$, RHS's $\mathtt{R2}$, β and the incidence matrix.

2. From α and `R2` combined, take the computation forward to give `R3`, the subset of these results that are consistent with, i.e subsets of, `R1`, if any.

3. If there are any such `R3`'s, for each case for the corresponding `R2` and set of values of β from the incidence matrix, combine α and the values of β and `O2` to give the set of CS's `Y`. For each such CS `Y`, carry out the backward search algorithm to obtain `O3` such that it is a subset of i.e. compatible with `O1` giving $\mathtt{A}' = \mathtt{O3} \rightarrow\rightarrow \mathtt{R3} \in \mathtt{IRR}(\mathtt{n}, \mathtt{k}+1)$ which is an extension of `A`, otherwise stop because there are no results.

4. Apply `F` to $\mathtt{A}'$ by starting the backward search from the original end point giving new abbreviated origins if any, RHS's i.e. members of $\mathtt{IRR}(\mathtt{n}+1, \mathtt{k}+1)$ and exceptions where the pointer goes away from the α, if any.

A further practical complication is that it is quicker to carry out these calculations for the same `O1` with different `R1`s (this is possible because of different symbols that are abbreviated in ...) together, rather than doing them all separately because the calculations with the `O1` are the same.

Table 16: Origins of IRR($\mathtt{n}+1$) derived from origins of IRR($\mathtt{n}$) and α, and exceptions for β

Origins and RHS's of IRR(n)	β and incidence matrix	Origins of IRR($\mathtt{n}+1$) for symbol α: a	b	c	d	e	β
2adβ... 2_aec...	b c 1 1		1dda... 1eda...				bc
2aeβ... 2_aec...	b c 1 1		4bea...	3aea...			c
2cdβ... 5_cca... 5_ccb... 5_cec...	a b c d 1 1 0 1 0 1 0 1 0 1 1 0	4ecc... 4eec... 5cad... 5ead...	1daa... 1dab... 1eaa... 1eab... 3ead... 4bcd...	3acd... 4cad...			abcd
2ceβ... 5_cca... 5_ccb... 5_cec...	a b c d 1 1 0 1 0 1 0 1 0 1 1 0						
2ddβ... 3_bca... 3_bcb...	a b d 1 1 1 0 1 1		1dca... 1eca... 4bcb... 4bcc...	3acb... 3acc...			abd
2deβ... 3_bca... 3_bcb...	a b d 1 1 1 0 1 1						
3abβ... 3_bcc...	b e 1 1						
3acβ... 2_abc... 3_bcc... 5_ccc... 5_cec...	a b c d e 1 1 1 0 1 0 0 0 0 1 0 0 0 1 0 0 0 0 1 0						ade
3aeβ... 2_aae...	a b 1 1		4beb...	3aeb...			

3$\underline{\text{e}}$aβ...	b c d e		
3_ecc...	0 0 1 0		
3_ece...	0 0 1 0		
4_cab...	0 1 0 1		
4_cae...	1 0 0 0		
4_cbc...	1 1 0 0		
4$\underline{\text{b}}$cβ...	a b c d e	2$\underline{\text{d}}$db... 2$\underline{\text{d}}$eb... 5$\underline{\text{c}}$eb... 5$\underline{\text{e}}$eb... 3$\underline{\text{e}}$eb... 2$\underline{\text{a}}$db... 2$\underline{\text{a}}$eb... 4$\underline{\text{c}}$eb... 2$\underline{\text{c}}$db... 2$\underline{\text{c}}$eb...	abcde
2_abc...	0 0 0 0 0		
3_ecc...	0 0 0 1 0		
3_ece...	0 0 0 1 0		
4_cab...	1 0 0 0 0		
4_cae...	0 0 0 0 1		
4_cbc...	1 1 1 0 1		
4$\underline{\text{c}}$aβ...			c
4$\underline{\text{e}}$cβ...			ac
1...βd$\underline{\text{b}}$	a b d		b
1...aba_	0 0 1		
1...cbd_	0 1 0		
2...bcb_	1 1 0		
2...bdb_	1 1 1		
2...cdb_	0 1 1		
1...βc$\underline{\text{c}}$	b e		b
1...abc_	1 1		
2...bdb_	1 1		
3...abc_	1 1		
3...dbc_	1 0		
1...βd$\underline{\text{c}}$	a b d		b
1...abc_	0 0 1		
2...bab_	1 1 1		
2...bdb_	1 1 0		
3...abc_	0 1 0		
4...aac_	0 1 0		
4...ccc_	0 0 1		
2...βb$\underline{\text{e}}$	a d		ad
1...abd_	1 1		
1...dbd_	0 1		
2...bab_	1 0		
2...cab_	0 1		
3...βa$\underline{\text{b}}$	b c d		
1...abd_	1 0 1		
1...dbd_	0 1 1		
2...bab_	1 1 1		
3...cbc_	0 1 1		
3...βb$\underline{\text{c}}$	a d	3...ee$\underline{\text{c}}$ 1...ee$\underline{\text{a}}$ 5...ee$\underline{\text{a}}$	a
1...abc_	1 1		
1...dbc_	0 1		
2...bab_	1 1		
4...bcc_	1 0		
4...ccc_	0 1		
3...βa$\underline{\text{d}}$	b e		
1...abc_	1 1		
1...dbd_	1 1		
3...βb$\underline{\text{d}}$	b c d		bc
1...abc_	1 0 1		
1...cbd_	0 1 1		
1...dbd_	1 0 1		
3...caa_	0 1 1		

4...βb$\underline{\texttt{a}}$ 2...bab_ 2...cab_ 3...abc_ 3...dbc_	a d 1 0 0 1 1 1 0 1	3...bd$\underline{\texttt{d}}$ 4...bd$\underline{\texttt{d}}$	
4...βc$\underline{\texttt{a}}$ 2...bdb_ 3...abc_ 4...acb_ 4...ccb_	b e 1 1 1 1 1 0 1 1		b
4...βb$\underline{\texttt{d}}$ 1...abc_ 1...cbd_ 1...dbd_ 3...caa_	b c d 1 0 1 0 1 1 1 0 1 0 1 1		bc
5...βa$\underline{\texttt{d}}$ 1...abc_ 1...dbc_ 3...abc_ 4...bcc_	b c d 1 0 1 0 1 1 0 1 1 1 1 1		bc

Table 17: RHS's of IRR($\mathtt{n}+\mathtt{1}$) derived from RHS's of IRR($\mathtt{n}$) and α

RHS of IRR($\mathtt{n}$)	RHS of IRR($\mathtt{n}+\mathtt{1}$) for symbol α				
	a	b	c	d	e
2_aae...		4_caa...	2abc...		
2_abc...	2dba...	4_cab...	2aba...	2cba...	3_cab...
2_aec...	5_ccc...	4_cae...	3_bcc...		
3_bca...		4_cbc...	2_abc...		
3_bcb...		4_cbc...	2_abc...		
3_ecc...	2cad...	4_cec...	2_aec...	5_cec...	
3_ece...	3caa...	4_cec...	2_aec...	5_cec...	5bcc...
4_cab...	3_bca...	4bcc...	2abd...	5_cca...	
4_cae...	3_bca...	5bcc...	2abc...	5_cca...	
4_cbc...	3_bcb...	1bab...	2aba...	5_ccb...	
5_cca...	2_ecc...	3_ecc...	2dba...		
5_ccb...	2_ecc...	3_ecc...	5_cec...		
5_ccc...	2_ecc...	3_ecc...	5_cec...	4_acc...	5_cec...
5_cec...	2_ece...	3_ece...	5_ccc...		
1...abc_	2...bdb_			2...bcb_	2...bcb_
1...dbc_	2...bdb_			2...bcb_	2...bcb_
2...bab_	1...abc_		1...abd_	1...aba_	4...bcc
2...cab_			1...abd_		
3...abc_			2...bab_		
3...dbc_			2...bab_		
4...bcc_	3...abc_			2...cdb_	5...cca_
4...ccc_	3...abc_			2...cdb_	5...cca_

Table 18: Origins of IRR($\mathtt{n}+\mathtt{1}$) derived from origins of IRR($\mathtt{n}$) for $\mathtt{k} = \mathtt{3}$ exceptions

Origin of IRR($\mathtt{n}$)	Origins of IRR($\mathtt{n}+\mathtt{1}$)
$\mathtt{2\underline{a}db}\ldots$	$\mathtt{4}\alpha\mathtt{dc\underline{a}}\ldots$
$\mathtt{2\underline{a}dc}\ldots$	$\mathtt{2}\alpha\mathtt{dc\underline{e}}\ldots$
$\mathtt{2\underline{a}ec}\ldots$	$\mathtt{3}\alpha\mathtt{aa\underline{d}}\ldots$,$\mathtt{4}\alpha\mathtt{aa\underline{d}}\ldots$ $\mathtt{3}\alpha\mathtt{ed\underline{d}}\ldots$,$\mathtt{4}\alpha\mathtt{ed\underline{d}}\ldots$
$\mathtt{2\underline{c}da}\ldots$	$\mathtt{1}\alpha\mathtt{ac\underline{c}}\ldots$,$\mathtt{1}\alpha\mathtt{cd\underline{c}}\ldots$ $\mathtt{5}\alpha\mathtt{ed\underline{d}}\ldots$,$\mathtt{1}\alpha\mathtt{ed\underline{c}}\ldots$ $\mathtt{5}\alpha\mathtt{cd\underline{d}}\ldots$
$\mathtt{2\underline{c}db}\ldots$	$\mathtt{4}\alpha\mathtt{ac\underline{a}}\ldots$,$\mathtt{4}\alpha\mathtt{ed\underline{a}}\ldots$,$\mathtt{4}\alpha\mathtt{cd\underline{a}}\ldots$
$\mathtt{2\underline{c}dc}\ldots$	$\mathtt{2}\alpha\mathtt{cd\underline{b}}\ldots$,$\mathtt{3}\alpha\mathtt{cd\underline{b}}\ldots$ $\mathtt{2}\alpha\mathtt{ed\underline{e}}\ldots$,$\mathtt{2}\alpha\mathtt{ed\underline{b}}\ldots$ $\mathtt{3}\alpha\mathtt{ed\underline{b}}\ldots$,$\mathtt{2}\alpha\mathtt{ac\underline{e}}\ldots$ $\mathtt{2}\alpha\mathtt{cd\underline{e}}\ldots$
$\mathtt{2\underline{c}dd}\ldots$	$\mathtt{1}\alpha\mathtt{cd\underline{b}}\ldots$,$\mathtt{1}\alpha\mathtt{ed\underline{b}}\ldots$
$\mathtt{2\underline{d}da}\ldots$	$\mathtt{1}\alpha\mathtt{cb\underline{c}}\ldots$,$\mathtt{3}\alpha\mathtt{cb\underline{c}}\ldots$,$\mathtt{1}\alpha\mathtt{cc\underline{c}}\ldots$
$\mathtt{2\underline{d}db}\ldots$	$\mathtt{4}\alpha\mathtt{cc\underline{a}}\ldots$,$\mathtt{4}\alpha\mathtt{cb\underline{a}}\ldots$
$\mathtt{2\underline{d}dd}\ldots$	$\mathtt{1}\alpha\mathtt{cb\underline{a}}\ldots$
$\mathtt{3\underline{a}ca}\ldots$	$\mathtt{3}\alpha\mathtt{ae\underline{c}}\ldots$
$\mathtt{3\underline{a}cd}\ldots$	$\mathtt{1}\alpha\mathtt{ae\underline{a}}\ldots$
$\mathtt{3\underline{a}ce}\ldots$	$\mathtt{5}\alpha\mathtt{ae\underline{a}}\ldots$
$\mathtt{4\underline{b}ca}\ldots$	$\mathtt{1}\alpha\mathtt{bb\underline{c}}\ldots$ $\mathtt{5}\alpha\mathtt{ea\underline{d}}\ldots$
$\mathtt{4\underline{b}cb}\ldots$	$\mathtt{4}\alpha\mathtt{bb\underline{a}}\ldots$
$\mathtt{4\underline{b}cc}\ldots$	$\mathtt{2}\alpha\mathtt{bb\underline{e}}\ldots$,$\mathtt{2}\alpha\mathtt{ea\underline{b}}\ldots$,$\mathtt{3}\alpha\mathtt{ea\underline{b}}\ldots$
$\mathtt{4\underline{b}cd}\ldots$	$\mathtt{1}\alpha\mathtt{bb\underline{a}}\ldots$,$\mathtt{1}\alpha\mathtt{ea\underline{b}}\ldots$
$\mathtt{4\underline{b}ce}\ldots$	$\mathtt{5}\alpha\mathtt{bb\underline{a}}\ldots$,$\mathtt{5}\alpha\mathtt{bb\underline{b}}\ldots$
$\mathtt{4\underline{c}ac}\ldots$	$\mathtt{3}\alpha\mathtt{cd\underline{d}}\ldots$,$\mathtt{4}\alpha\mathtt{cd\underline{d}}\ldots$
$\mathtt{4\underline{e}ca}\ldots$	$\mathtt{3}\alpha\mathtt{eb\underline{c}}\ldots$,$\mathtt{1}\alpha\mathtt{eb\underline{c}}\ldots$
$\mathtt{4\underline{e}cc}\ldots$	$\mathtt{2}\alpha\mathtt{eb\underline{e}}\ldots$
$\mathtt{1}\ldots\mathtt{bd\underline{b}}$	$\mathtt{1}\ldots\mathtt{\underline{d}cb}\alpha$,$\mathtt{1}\ldots\mathtt{\underline{e}cb}\alpha$
$\mathtt{1}\ldots\mathtt{bc\underline{c}}$	$\mathtt{1}\ldots\mathtt{\underline{d}ac}\alpha$,$\mathtt{1}\ldots\mathtt{\underline{e}ac}\alpha$
$\mathtt{1}\ldots\mathtt{bd\underline{c}}$	$\mathtt{1}\ldots\mathtt{\underline{d}cc}\alpha$,$\mathtt{1}\ldots\mathtt{\underline{e}cc}\alpha$
$\mathtt{2}\ldots\mathtt{ab\underline{e}}$	$\mathtt{2}\ldots\mathtt{\underline{d}de}\alpha$,$\mathtt{2}\ldots\mathtt{\underline{d}ee}\alpha$
$\mathtt{2}\ldots\mathtt{db\underline{e}}$	$\mathtt{2}\ldots\mathtt{\underline{c}de}\alpha$,$\mathtt{2}\ldots\mathtt{\underline{c}ee}\alpha$
$\mathtt{3}\ldots\mathtt{ab\underline{c}}$	$\mathtt{5}\ldots\mathtt{\underline{c}ec}\alpha$,$\mathtt{5}\ldots\mathtt{\underline{e}ec}\alpha$
$\mathtt{3}\ldots\mathtt{bb\underline{d}}$	$\mathtt{3}\ldots\mathtt{\underline{e}ed}\alpha$
$\mathtt{3}\ldots\mathtt{cb\underline{d}}$	$\mathtt{4}\ldots\mathtt{\underline{c}ed}\alpha$
$\mathtt{4}\ldots\mathtt{bc\underline{a}}$	$\mathtt{1}\ldots\mathtt{\underline{d}dc}\alpha$,$\mathtt{1}\ldots\mathtt{\underline{e}dc}\alpha$,$\mathtt{3}\ldots\mathtt{\underline{e}aa}\alpha$
$\mathtt{4}\ldots\mathtt{bb\underline{d}}$	$\mathtt{4}\ldots\mathtt{\underline{b}bd}\alpha$
$\mathtt{4}\ldots\mathtt{cb\underline{d}}$	$\mathtt{3}\ldots\mathtt{\underline{a}bd}\alpha$
$\mathtt{5}\ldots\mathtt{ba\underline{d}}$	$\mathtt{4}\ldots\mathtt{\underline{b}ed}\alpha$
$\mathtt{5}\ldots\mathtt{ca\underline{d}}$	$\mathtt{3}\ldots\mathtt{\underline{a}ed}\alpha$

Table 16 represents in minimal form some of the rules by which IRR can be changed to new IRR of length $\mathtt{n}$ greater by 1. These rules are of length $\mathtt{k} = \mathtt{3}$ because they can be written with 3 consecutive symbols. Related information such as the new RHS's, which cannnot be fitted into Table 16 without repetition are given in Table 17. The exceptions where the pointer does not go in the expected direction in the derivation of new origins are given in Table 18 and these are also indicated in the rightmost column of Table 16 where the appropriate values of β are given as was done in Table 13. The aim of all these results is to represent $\mathtt{F}$ (i.e. the set of all rules by which all the IRR can be generated starting from all the IRR of length 1, the single TM steps) in terms particular to the TM being studied, in a finite form. There may be a finite or an infinite number of rules in $\mathtt{F}$, but in the latter case it is hoped that it can always be represented finitely.

Table 19: Origins of IRR($\mathtt{n}+\mathtt{1}$) derived from origins of IRR($\mathtt{n}$) for $\mathtt{k} = \mathtt{4}$

Origin of IRR($\mathtt{n}$)	Origins of IRR($\mathtt{n}+\mathtt{1}$)
$\mathtt{2}\alpha\underline{\mathtt{a}}\mathtt{dbd}\ldots$	$\mathtt{1}\alpha\mathtt{dca}\underline{\mathtt{b}}\ldots$

11 Reverse rules

By following the backward searching algorithm starting from a CS on the LHS, a branching tree is in general produced, and this can be summarised by omitting all intermediate results, just giving the final set of CS's (origins) on the RHS of the reverse rule. This gives a reverse rule which is defined to have a CS the LHS and a set of CS's on the RHS. The set of CS's on the RHS of a reverse rule is the set of all possible origins for the LHS, which are traced back as far as possible but ignoring branches that terminate with the pointer not at either end of the string (condition 3). In general, the CS's on the RHS of a reverse rule will have some with the pointer at the right, and others with the pointer at the left.

An irreducible reverse rule is a reverse rule that has no redundant symbols in it. This is analogous to the usage for (forward) computation rules. The length of a reverse rule is the number of symbols in the LHS and each of its origin CS's in the RHS.

Let $\mathtt{CS}(\mathtt{n},\mathtt{p})$ be the set of all CS's such that $\mathtt{n}$ and $\mathtt{p}$ are respectively the length and position of the pointer (from 1 at the left to $\mathtt{n}$ at the right) in the CS. Extra symbols can be added at either end to generate other sets of CS's in the obvious manner, so for example $\alpha\mathtt{CS}(\mathtt{n},\mathtt{1})$ is the set obtained from $\mathtt{CS}(\mathtt{n},\mathtt{1})$ by adding the symbol α on the left to each member of $\mathtt{CS}(\mathtt{n},\mathtt{1})$, so the pointer is now immediately to the right of the α i.e. at position 2 in the string for each member.

The truncation of a CS $\mathtt{X}$ of length $\mathtt{n}$ to length $\mathtt{l}$ is defined when $\mathtt{l} \leq \mathtt{n}$ and is formed by truncating the string to length $\mathtt{l}$ so that any symbols lost are taken from the end of the string furthest from the pointer. Ambiguity will only arise if the pointer is in the middle of the string. An alternative terminology will be used if needed in case of ambiguity.

Backward searching to find the origins of a CS $\mathtt{X}$ in $\mathtt{CS}(\mathtt{n},\mathtt{n})$ (respectively $\mathtt{CS}(\mathtt{n},\mathtt{1})$) leads in general to two sets of CS's, one set denoted by $\mathtt{O}_1(\mathtt{X})$ are each in $\mathtt{CS}(\mathtt{n},\mathtt{1})$ (respectively $\mathtt{CS}(\mathtt{n},\mathtt{n})$) ending in condition 1 and the other set denoted by $\mathtt{O}_2(\mathtt{X})$ are each in $\mathtt{CS}(\mathtt{n},\mathtt{n})$ (respectively $\mathtt{CS}(\mathtt{n},\mathtt{1})$) ending in condition 2. The reverse rule (RevR to distinguish it from RR which is a regular rule) derived is written as

$$\mathtt{X} \leftarrow \begin{cases} \mathtt{O}_1(\mathtt{X}) \text{ (condition 1)} \\ \mathtt{O}_2(\mathtt{X}) \text{ (condition 2)} \end{cases} . \tag{610}$$

Reverse rules result from derivations of the reachability of their LHS's. A CS $\mathtt{X}$ is defined to be reachable if and only if $\mathtt{O}_1(\mathtt{X}) \neq \emptyset$. This is involved in deriving the IRR($\mathtt{n}$) for a TM. The backward searching algorithm and the 3 conditions any branch can end in was described in detail in my earlier paper [2] section 2.2. The branches in $\mathtt{O}_2(\mathtt{X})$ cannot lead to a proof of the reachability of $\mathtt{X}$. If $\mathtt{X}$ is the LHS of a member of IRR($\mathtt{n}$) of type RL or LR, there is a subset $\mathtt{S}$ of the LHS's of IRR($\mathtt{n}+1$) associated with $\mathtt{X}$ that can be obtained from $\mathtt{X}$ by firstly adding an arbitrary single symbol (here called α) at the opposite end of the string from the pointer in $\mathtt{X}$, and continuing the backward search algorithm applied to $\mathtt{X}$ to find the set $\mathtt{S}$ of $\alpha\mathtt{X}$ that are reachable i.e. such that $\mathtt{O}_1(\alpha\mathtt{X}) \neq \emptyset$. The pointer is assumed to be at the right in $\mathtt{X}$ in this case. Otherwise α would be added on the right and $\mathtt{S}$ would be $\{\mathtt{X}\alpha\}|\mathtt{O}_1(\mathtt{X}\alpha) \neq \mathtt{0}$

The backward search applied to $\alpha\mathtt{X}$ must start as in (610) with α on the left which plays no role unless the pointer reaches position 2. If this does happen (result in $\alpha\mathtt{O}_1(\mathtt{X})$) the pointer could finally reach position 1 without first getting to position $\mathtt{n}+1$ (result in $\mathtt{O}_1(\alpha\mathtt{X})$) or to position $\mathtt{n}+1$ without first getting to position 1. In the latter case the result is in $\mathtt{O}_2(\alpha\mathtt{X})$ but not in $\alpha\mathtt{O}_2(\mathtt{X})$. $\alpha\mathtt{O}_2(\mathtt{X})$ is reached without the pointer ever getting to position 2 and not otherwise. This is expressed as follows:

$$\alpha\mathtt{X} \leftarrow \begin{cases} \alpha\mathtt{O}_1(\mathtt{X}) \leftarrow \begin{cases} \mathtt{O}_1(\alpha\mathtt{X}) \\ \mathtt{O}_2(\alpha\mathtt{X}) \setminus \alpha\mathtt{O}_2(\mathtt{X}) \end{cases} \\ \alpha\mathtt{O}_2(\mathtt{X}) \end{cases} \tag{611}$$

Note that it is not possible for the same RHS to be obtained in (610) from two different paths because the forward computation is unique. In $\alpha\mathtt{O}_2(\mathtt{X})$ the pointer is at the right hand end of the string, so the backward search terminates. This shows that in (610), the extension of the derivation of the reachability of $\mathtt{X}$ to that of $\alpha\mathtt{X}$, only the $\mathtt{O}_1$ branch of the former is needed.

Involved in (611) is the backward searching algorithm applied to CS's of the form $\mathtt{CS}(\mathtt{n},2)$ ($\alpha\mathtt{O}_1(\mathtt{X})$ with $\mathtt{n}$ replaced by $\mathtt{n}-1$). The result of this is

$$\alpha\mathtt{O}_1(\mathtt{X}) \leftarrow \begin{cases} \mathtt{O}_1(\alpha\mathtt{X}) \\ \mathtt{O}_2(\alpha\mathtt{X}) \setminus \alpha\mathtt{O}_2(\mathtt{X}) \end{cases} \tag{612}$$

This will be called an auxiliary reverse rule (ARR) to make the distinction between these and (final) reverse rules such as (610). In general an ARR of length $\mathtt{n}$ will be a reverse rule having a LHS in $\mathtt{CS}(\mathtt{n},2)$ or $\mathtt{CS}(\mathtt{n},\mathtt{n}-1)$. All types of reverse rules have an RHS which consists of two sets of CS's, which are the sets of endpoints of the backward search algorithm, so the pointer is at one end of the string (condition 3 branches are deleted). They will be denoted by the functions $\mathtt{O}_1$ and $\mathtt{O}_2$ applied to the LHS as in (610) as if the LHS had had the pointer at the adjacent endpoint of the string. This definition only fails if the length $\mathtt{n}$ of the ARR is $\leq \mathtt{3}$. The notations $\mathtt{O}_\mathtt{L}(\mathtt{X})$ and $\mathtt{O}_\mathtt{R}(\mathtt{X})$ will likewise refer to the sets of origins of $\mathtt{X}$ with the pointer at the left and right respectively. These can be used whenever $\mathtt{n} \geq \mathtt{1}$ (but it is trivial if $\mathtt{n} = \mathtt{1}$ because then $\mathtt{O}_\mathtt{L}(\mathtt{X}) = \mathtt{O}_\mathtt{R}(\mathtt{X}) = \mathtt{X}$).

Now it is possible to describe in general the application of an ARR

$$\mathtt{Z} \leftarrow \{\mathtt{O}_1(\mathtt{Z}), \mathtt{O}_2(\mathtt{Z})\} \tag{613}$$

to extending the reachability of $\mathtt{X}$ indicated by

$$\mathtt{X} \leftarrow \mathtt{Y} \tag{614}$$

for some CS's $\mathtt{X} \in \mathtt{CS}(\mathtt{n}, \mathtt{n})$ and $\mathtt{Y} \in \mathtt{CS}(\mathtt{n}, \mathtt{1})$. In this case the α must be added on the left giving

$$\alpha\mathtt{X} \leftarrow \alpha\mathtt{Y}. \tag{615}$$

Continuing the backward search algorithm gives

$$\alpha\mathtt{X} \leftarrow \alpha\mathtt{Y} \leftarrow \{\mathtt{O}_1(\alpha\mathtt{Y}), \mathtt{O}_2(\alpha\mathtt{Y})\} \tag{616}$$

where by definition $\mathtt{O}_1(\alpha\mathtt{Y}) = \mathtt{O}_2(\alpha\mathtt{X})$ and $\mathtt{O}_2(\alpha\mathtt{Y}) = \mathtt{O}_1(\alpha\mathtt{X})$. If $\alpha\mathtt{Y}$ matches $\mathtt{Z}$ i.e. $\alpha\mathtt{Y} = \mathtt{ZT}$ for some string $\mathtt{T}$, and using (613) gives

$$\alpha\mathtt{X} \leftarrow \alpha\mathtt{Y} = \mathtt{ZT} \leftarrow \{\mathtt{O}_1(\mathtt{Z})\mathtt{T} = \mathtt{O}_1(\alpha\mathtt{Y}), \mathtt{O}_2(\mathtt{Z})\mathtt{T} = \mathtt{O}_2(\alpha\mathtt{Y})\}. \tag{617}$$

Here $\alpha\mathtt{Y}$ has the pointer at position 2 so $\mathtt{O}_2(\alpha\mathtt{Y})$ has the pointer at position 1 therefore the $\mathtt{O}_2(\alpha\mathtt{Y})$ branch demonstrates the reachability of $\alpha\mathtt{X}$. That is, the reachability of $\alpha\mathtt{X}$ true if and only if $\mathtt{O}_2(\alpha\mathtt{Y}) \neq \emptyset$ ($\Leftrightarrow \mathtt{O}_2(\mathtt{Z}) \neq \emptyset$). If $\mathtt{O}_1(\alpha\mathtt{Y}) = \emptyset$ ($\Leftrightarrow \mathtt{O}_1(\mathtt{Z}) = \emptyset$), the pointer cannot reach the right hand end in (616). If the rightmost position of the pointer is $\mathtt{l} \leq \mathtt{n}$, (613) can be truncated from the right to length $\mathtt{l}+\mathtt{1}$ (if it was truncated to length $\mathtt{l}$ and the backward searching algorithm would stop with the pointer at the right) and then $\mathtt{T}$ has length $\mathtt{n}-\mathtt{l}$. Otherwise $\mathtt{T}$ is the empty string ϵ of length 0 and there is no truncation in (616). The ARR (613) will be called irreducible because there are then no redundant symbols in it and (613) will be referred to as an AIRR (auxiliary irreducible reverse rule) in analogy with the irreducible regular rules (IRR) introduced earlier [1] and will be unique. From now on any ARR applied to continue the derivation of the reachability of $\mathtt{X}$ to that of $\alpha\mathtt{X}$ (or $\mathtt{X}\alpha$) will be assumed to be an AIRR.

The mirror image form when $\mathtt{X} \leftarrow \mathtt{Y}$ where $\mathtt{X} \in \mathtt{CS}(\mathtt{n}, \mathtt{1})$ and $\mathtt{Y} \in \mathtt{CS}(\mathtt{n}, \mathtt{n})$ is treated similarly with the matching condition being $\mathtt{Y}\alpha = \mathtt{TZ}$.

11.1 Classification of reverse rules

It has been found useful to classify an AIRR (613) (could also be applicable to RevR) according to whether $\mathtt{O}_1(\mathtt{X}) = \emptyset$ and whether $\mathtt{O}_2(\mathtt{X}) = \emptyset$ where $\mathtt{X}$ is its LHS.

If both $\mathtt{O}_1(\mathtt{X}) \neq \emptyset$ and $\mathtt{O}_2(\mathtt{X}) \neq \emptyset$, the AIRR has origins for the LHS with the pointer at either end of the string and will be called a two-way AIRR (written as $\pm$). This is true for most of the AIRR of length 3 and sometimes occurs for the AIRR of length 4 and 5 for TM1. This situation might indicate that the AIRR could be extended by adding an extra symbol on its LHS $\mathtt{X}$ (to the right if $\mathtt{X} \in \mathtt{CS}(2, \mathtt{n})$ and to the left if $\mathtt{X} \in \mathtt{CS}(\mathtt{n}, \mathtt{n}-1)$) and following through to generate the RHS such that the new AIRR is not of this type.

Now suppose $\mathtt{O}_1(\mathtt{X}) \neq \emptyset$ and $\mathtt{O}_2(\mathtt{X}) = \emptyset$. In this case all the origins $\mathtt{Y}$ of $\mathtt{X}$ have the pointer at the far end of the string from where it started i.e. $\mathtt{Y} \in \mathtt{CS}(\mathtt{n}, \mathtt{n})$ if $\mathtt{X} \in \mathtt{CS}(\mathtt{n}, 2)$ and $\mathtt{Y} \in \mathtt{CS}(\mathtt{n}, 1)$ if $\mathtt{X} \in \mathtt{CS}(\mathtt{n}, \mathtt{n}-1)$. These are the direction changing one-way AIRR (written as $-$).

Now consider the reverse i.e. $\mathtt{O}_1(\mathtt{X}) = \emptyset$ and $\mathtt{O}_2(\mathtt{X}) \neq \emptyset$. In this case the pointer starts at position 2 (respectively $\mathtt{n}-1$) and ends at position 1 (respectively $\mathtt{n}$) in all branches of the reverse computation. These are non direction changing one-way AIRR (written as $+$).

Finally if both $\mathtt{O}_1(\mathtt{X}) = \emptyset$ and $\mathtt{O}_2(\mathtt{X}) = \emptyset$ the AIRR terminates the search with no results and the AIRR may be called a null AIRR (written as $\emptyset$).

To classify a reverse rule, 3 elements will be used in this order: its length, the pointer position in its LHS, the set of pointer positions in the RHS's. This is done according to the following table:

Reverse rule type	Condition	Position of pointer in LHS	Symbol for LHS	Possible symbols for RHS
RevR	$\mathtt{n} \geq 2$	1	L	$\pm, +, -, \emptyset$
	$\mathtt{n} \geq 2$	$\mathtt{n}$	R	$\pm, +, -, \emptyset$
AIRR	$\mathtt{n} = 3$	2		$\pm$, $\mathtt{L}$, $\mathtt{R}$, $\emptyset$
	$\mathtt{n} > 3$	2	L	$\pm, +, -, \emptyset$
	$\mathtt{n} > 3$	$\mathtt{n}-1$	R	$\pm, +, -, \emptyset$

(618)

This gives the following possible classifications for AIRR: $\mathtt{3}\pm$, $\mathtt{3L}$, $\mathtt{3R}$, $\mathtt{3}\emptyset$, and $\mathtt{nL}\pm$, $\mathtt{nL}-$, $\mathtt{nL}+$, $\mathtt{nL}\emptyset$, $\mathtt{nR}\pm$, $\mathtt{nR}-$, $\mathtt{nR}+$, $\mathtt{nR}\emptyset$ where $\mathtt{n} \geq 4$, and the following for RevR: $\mathtt{nL}\pm$, $\mathtt{nL}-$, $\mathtt{nL}+$, $\mathtt{nL}\emptyset$, $\mathtt{nR}\pm$, $\mathtt{nR}-$, $\mathtt{nR}+$, $\mathtt{nR}\emptyset$ where $\mathtt{n} \geq 3$.

Therefore using this notation, the result above can be written as follows:

Lemma 11.1. *Given a reachability derivation* $\mathtt{X} \leftarrow \mathtt{Y}$ *for* $\mathtt{X} \in \mathtt{CS}(\mathtt{n}, \mathtt{n})$ *(respectively* $\mathtt{X} \in \mathtt{CS}(\mathtt{n}, 1)$*), for which necessarily* $\mathtt{Y} \in \mathtt{CS}(\mathtt{n}, 1)$ *(respectively* $\mathtt{Y} \in \mathtt{CS}(\mathtt{n}, \mathtt{n})$*), then an AIRR* $\mathtt{A}$ *of the form* (610) *of length* $\leq \mathtt{n}+1$ *can be found uniquely such that when the substitution* (617) *is made (respectively using* $\mathtt{Y}\alpha = \mathtt{ZT}$*), the reverse rule generated by* $\alpha\mathtt{X}$ *(respectively* $\mathtt{X}\alpha$*) is produced,* *and* $\alpha\mathtt{X}$ *(respectively* $\mathtt{X}\alpha$*) is reachable if and only if* $\mathtt{O}_2(\alpha\mathtt{Y}) \neq \emptyset$ *(respectively* $\mathtt{O}_2(\mathtt{Y}\alpha) \neq \emptyset$*). If the length of* $\mathtt{C}$ *is less than* $\mathtt{n}+\mathtt{1}$ *(which is true if and only if* $\mathtt{O}_1(\alpha\mathtt{Y}) = \emptyset$ *or respectively* $\mathtt{O}_1(\mathtt{Y}\alpha) = \emptyset$*) then the type of* $\mathtt{A}$ *is* $+$ *or* $\emptyset$*. Note the extension of the definition of reachability to* $\mathtt{CS}(\mathtt{n},2)$ *and* $\mathtt{CS}(\mathtt{n},\mathtt{n}-\mathtt{1})$*. This was done so that Lemma 11.1 can be extended to starting with an AIRR.*

The matching in (617) suggests that $\mathtt{Z}$ should have its leftmost symbol arbitrary, for application to the case when $\mathtt{X} \in \mathtt{CS}(\mathtt{n},\mathtt{n})$ and $\mathtt{Y} \in \mathtt{CS}(\mathtt{n},1)$. Likewise so should be its rightmost symbol to allow the AIRR to be used in the same way if the α was added on the right. Thus a parameterised set of AIRR's with an arbitrary symbol on the left or on the right or both should probably be considered as a single AIRR.

Unless any range of values of any placeholder symbol is specified, it will be assumed to be the complete set of symbols used by the TM under discussion. Many equations in this paper define parameterised sets of reverse rules. Sometimes placeholder symbols can appear in the RHS but not in the LHS of a reverse rule. This happens when the results in the RHS are summarised. Then the range of the placeholder symbol needs to be specified.

Now suppose instead of the conditions of Lemma 11.1, $\mathtt{X} \in \mathtt{CS}(\mathtt{n},\mathtt{n}-\mathtt{1})$ (respectively $\mathtt{X} \in \mathtt{CS}(\mathtt{n},2)$) so that one branch of an AIRR is under consideration i.e. $\mathtt{X} \leftarrow \mathtt{Y}$ where $\mathtt{Y} \in \mathtt{CS}(\mathtt{n},1)$ (respectively $\mathtt{Y} \in \mathtt{CS}(\mathtt{n},\mathtt{n})$). Then because $\mathtt{X}$ plays very little role in the arguments, the result is the same except that the concept of reachability has been extended to CS's $\in \mathtt{CS}(\mathtt{n},2) \cup \mathtt{CS}(\mathtt{n},\mathtt{n}-\mathtt{1})$. The conclusion is that there is a unique AIRR $\mathtt{A}$ (613) of length $\leq \mathtt{n}+\mathtt{1}$ such that when the substitution (617) (or its mirror image form) is made the reverse rule generated by the extension of $\mathtt{X}$, $\alpha\mathtt{X}$ (respectively $\mathtt{X}\alpha$) is produced and this extension of $\mathtt{X}$ is reachable if and only if $\mathtt{O}_2(\alpha\mathtt{Y}) \neq \emptyset$ (respectively $\mathtt{O}_2(\mathtt{Y}\alpha) \neq \emptyset$). Also the length of the AIRR $\mathtt{A}$ is $< \mathtt{n}+\mathtt{1}$ if and only if $\mathtt{O}_1(\alpha\mathtt{Y}) = \emptyset$ (respectively $\mathtt{O}_1(\mathtt{Y}\alpha) = \emptyset$), and this implies that the type of $\mathtt{C}$ is $+$ or $\emptyset$.

I shall now take the further obvious step of extending the hypothesis to allow multiple RHS's i.e. assuming an AIRR, say $\mathtt{C}$ whose length is ≥ 4. The pointer in the LHS of $\mathtt{C}$ is next to the opposite end of the string from where it is in an $\mathtt{O}_1$ branch of $\mathtt{C}$, and the α is at this end too and the substitution (613) can be made, however for the RHS's that are in $\mathtt{O}_2$, the α is at the opposite end from where the pointer is when a substitution is to be made, and so the backward search terminates. Thus the above arguments only apply to the $\mathtt{O}_1$ branches of $\mathtt{C}$. The result is now stated as a theorem because of its obvious importance.

Theorem 11.2. *Suppose that* $\mathtt{C}$ *is an AIRR or an RevR of length* $\mathtt{n} \geq 4$ *for the TM under consideration so that* $\mathtt{C}$ *can be expressed by* (613)*. Now add the arbitrary symbol* α *(on the left or the right) to the LHS and each RHS of* $\mathtt{C}$ *giving say* $\mathtt{B}$*. Whether the added symbol is on the left or right is determined such that the LHS of* $\mathtt{B} \in \mathtt{CS}(\mathtt{n}+1,2) \cup \mathtt{CS}(\mathtt{n}+1,\mathtt{n}) \cup \mathtt{CS}(\mathtt{n}+1,1) \cup \mathtt{CS}(\mathtt{n}+1,\mathtt{n}+1)$*. Then* $\mathtt{B}$ *follows from* $\mathtt{C}$*, and the backward searching algorithm can be applied to* *each RHS of* $\mathtt{B}$ *that arose from an* $\mathtt{O}_1$ *branch of* $\mathtt{C}$*. For each case this leads to a unique AIRR again of the form* $\mathtt{C}$ (613) *of length* $\leq \mathtt{n}+1$ *such that when the substitutions* (617) *(or its mirror image form) are made for each* $\mathtt{O}_1$ *branch of* $\mathtt{C}$*, the reverse rule* $\mathtt{D}$ *generated by the LHS of* $\mathtt{B}$ *is produced and the LHS of* $\mathtt{B}$ *(= LHS of* $\mathtt{D}$*) is reachable if and only if there exists an RHS* $\mathtt{E}$ *of* $\mathtt{B}$ *such that* $\mathtt{O}_2(\mathtt{E}) \neq \emptyset$*. For each such AIRR* $\mathtt{A}$*, the length of* $\mathtt{C} < \mathtt{n}+1$ *if and only if* $\mathtt{O}_1(\mathtt{E}) = \emptyset$*, and this implies that the type of* $\mathtt{C}$ *is* $\neg\mathtt{M}$*.*

Similarly to (611), if the LHS of $\mathtt{C}$ is $\mathtt{Z} \in \mathtt{CS}(\mathtt{n},2)$ then the final reverse rule say $\mathtt{D}$ can be represented as follows showing the intermediate steps:

$$\mathtt{Z}\alpha \leftarrow \begin{cases} \mathtt{O}_1(\mathtt{Z})\alpha \leftarrow \begin{cases} \mathtt{O}_1(\mathtt{Z}\alpha) \\ \mathtt{O}_2(\mathtt{Z}\alpha) \setminus \mathtt{O}_2(\mathtt{Z})\alpha \end{cases} \\ \mathtt{O}_2(\mathtt{Z})\alpha \end{cases} \tag{619}$$

Thus associated with the AIRR $\mathtt{C}$ and the symbol α at the start, there is a set of AIRR's generically called $\mathtt{C}$, and the result of the substitutions is the AIRR $\mathtt{D}$. Notations might be useful for these, so I define the functions $\mathtt{f}$ and $\mathtt{inc}$ (abbreviation of increment) such that $\mathtt{f}(\mathtt{C},\alpha)$ is the set of AIRR's $\mathtt{A}$, and $\mathtt{inc}(\mathtt{C},\alpha) = \mathtt{D}$.

It will be also useful to have the combined types $\mathtt{P}$ for plus i.e. $+$ or $\pm$ which means there is a backward search with the pointer staying at the same end of the string, likewise $\mathtt{M}$ for minus will mean $-$ or $\pm$ i.e. there is a backward search with the pointer going to the opposite end of the string. Then we can write

$$\begin{aligned} \mathtt{P} &\Leftrightarrow \pm \vee + \\ \mathtt{M} &\Leftrightarrow \pm \vee - \\ \neg\, \mathtt{P} &\Leftrightarrow - \vee \emptyset \\ \neg\, \mathtt{M} &\Leftrightarrow + \vee \emptyset \end{aligned} \tag{620}$$

and

$$\begin{aligned} \emptyset &\Leftrightarrow \neg\, \mathtt{P} \wedge \neg\, \mathtt{M} \\ + &\Leftrightarrow \, \mathtt{P} \wedge \neg\, \mathtt{M} \\ - &\Leftrightarrow \neg\, \mathtt{P} \wedge \, \mathtt{M} \\ \pm &\Leftrightarrow \, \mathtt{P} \wedge \, \mathtt{M} \end{aligned} \tag{621}$$

where as usual $\neg$ means *not*, $\wedge$ means *and*, $\vee$ means *inclusive or*, and $\Leftrightarrow$ means *logical equivalence*, and the $\neg$ operators are applied before any others by default.

In general the type of the AIRR $\mathtt{D}$ obtained as in Theorem 11.2 by extending the AIRR $\mathtt{C}$ by a single extra symbol is of interest. The following arguments show how this can be done.

Define "$\mathtt{D}$ has type $\mathtt{P}$ via $\mathtt{O}_1$" to mean that $\mathtt{D}$ has type $\mathtt{P}$ (a reverse path exists ending with the pointer at the same end of the string as where it starts) and there exists one of the these reverse paths starting (going backwards) from an $\mathtt{O}_1$ path of $\mathtt{C}$. Likewise $\mathtt{P}$ via $\mathtt{O}_2$ is defined. Because a backward $\mathtt{P}$ path must start with either $\mathtt{O}_1$ or $\mathtt{O}_2$ (paths via $\mathtt{O}_1$ have the pointer reach next to the α and paths via $\mathtt{O}_2$ do not) it follows that for any $\mathtt{D}$, $\mathtt{P} \Leftrightarrow (\mathtt{P}\text{ via }\mathtt{O}_1) \vee (\mathtt{P}\text{ via }\mathtt{O}_2)$.

These cases can be illustrated (including case $\mathtt{M}$) as follows:

$$\left.\begin{array}{r} \alpha\mathtt{xx}\underline{\mathtt{x}}\mathtt{x} \\ \uparrow \\ \mathtt{C}\left\{\begin{array}{c} \alpha\underline{\mathtt{x}}\mathtt{xxx} \\ \uparrow \\ \alpha\mathtt{xxx}\underline{\mathtt{x}} \end{array}\right. \end{array}\right\} \mathtt{P}\text{ via }\mathtt{O}_1 \qquad \left.\begin{array}{r} \alpha\mathtt{xx}\underline{\mathtt{x}}\mathtt{x} \\ \uparrow \\ \mathtt{C}\left\{\begin{array}{c} \alpha\underline{\mathtt{x}}\mathtt{xxx} \\ \uparrow \\ \underline{\mathtt{x}}\mathtt{xxxx} \end{array}\right. \end{array}\right\} \mathtt{M}\text{ via }\mathtt{O}_1 \qquad \left.\begin{array}{c} \alpha\mathtt{xx}\underline{\mathtt{x}}\mathtt{x} \\ \uparrow \\ \alpha\mathtt{xxx}\underline{\mathtt{x}} \end{array}\right\} \mathtt{P}\text{ via }\mathtt{O}_2 \tag{622}$$

Of course the mirror image forms also apply.

For the case via $\mathtt{O}_2$, only type $\mathtt{P}$ and not $\mathtt{M}$ is possible because $\mathtt{O}_2$ takes the pointer back to the same end of the string, opposite to the α where the backward search terminates. Thus $\mathtt{M} \Leftrightarrow (\mathtt{M}\text{ via }\mathtt{O}_1) \vee (\mathtt{M}\text{ via }\mathtt{O}_2) \Leftrightarrow \mathtt{M}\text{ via }\mathtt{O}_1$.

Now consider the type of the substitution AIRR $\mathtt{C}$ associated with each case. In the case that $\mathtt{D}$ has type $\mathtt{P}$ via $\mathtt{O}_1$ the reverse path must end next to the same end of the string as where it started, so the final part after the substitution must take the pointer back again, so $\mathtt{C}$ must have type $\mathtt{M}$.

$$\mathtt{D}\text{ has type }\mathtt{P}\text{ via }\mathtt{O}_1 \Leftrightarrow \exists\text{ an }\mathtt{O}_1\text{ branch of }\mathtt{C}\text{ such that }\mathtt{C}\text{ has type }\mathtt{M} \tag{623}$$

and because branches starting with $\mathtt{O}_2$ terminate without a substitution,

$$\mathtt{D}\text{ has type }\mathtt{P}\text{ via }\mathtt{O}_2 \Leftrightarrow \exists\text{ an }\mathtt{O}_2\text{ branch of }\mathtt{C}. \tag{624}$$

As above, $\mathtt{D}$ has type $\mathtt{M}$ via $\mathtt{O}_2$ is impossible, and

$$\mathtt{D}\text{ has type }\mathtt{M}\text{ via }\mathtt{O}_1 \Leftrightarrow \exists\text{ an }\mathtt{O}_1\text{ branch of }\mathtt{C}\text{ such that }\mathtt{A}\text{ has type }\mathtt{P}. \tag{625}$$

These statements combine to give the following

$$\left.\begin{array}{l} \mathtt{D}\text{ has type }\mathtt{P} \Leftrightarrow \begin{array}{l} (\exists\text{ an }\mathtt{O}_1\text{ branch of }\mathtt{C}\text{ such that }\mathtt{A}\text{ has type }\mathtt{M}) \vee \\ (\exists\text{ an }\mathtt{O}_2\text{ branch of }\mathtt{C}) \end{array} \\ \\ \mathtt{D}\text{ has type }\mathtt{M} \Leftrightarrow \exists\text{ an }\mathtt{O}_1\text{ branch of }\mathtt{C}\text{ such that }\mathtt{A}\text{ has type }\mathtt{P} \end{array}\right\} \tag{626}$$

The results (626) can be combined with (621) showing how the type of $\mathtt{D}$ can be determined as one of $\pm, +, -, \emptyset$.

References

[1] Methods for Understanding Turing Machine Computations

[2] John Nixon Reverse engineering Turing Machines and the Collatz Conjecture

[3] The previous version in D of the computer program for analysis of Turing Machines

[4] SIAM Journal on Computing, 1 (2): 146–160, Depth-First Search and Linear Graph Algorithms doi:10.1137/0201010

[5] An implementation of Tarjan's strongly connected components algorithm in D

[6] draft of new program doing the computations in this paper

In this section the consequences of the backward search algorithm moving the pointer in the unexpected direction i.e. away from the symbol α will be considered. This happened in many of the derivations in Tables 2 and 22 and is summarised in the rightmost column of Table 13. Here the values of β giving rise to such a case are listed. To understand this consider the following reverse rule obtained when attempting another derivation in a way similar to (87).

$$\mathtt{4}\ldots\mathtt{c\underline{a}}\alpha\left\{\begin{array}{l} \stackrel{\alpha=\mathtt{a}}{\leftarrow}\mathtt{5ca\underline{d}} \\ \stackrel{\alpha=\mathtt{c}}{\leftarrow}\left\{\begin{array}{l}\mathtt{2ca\underline{b}}\\ \mathtt{3ca\underline{b}}\end{array}\right. \\ \stackrel{\alpha=\mathtt{d}}{\leftarrow}\mathtt{1ca\underline{b}} \\ \leftarrow\mathtt{3\underline{a}a}\alpha \end{array}\right. . \tag{627}$$

Following up just the single bottom branch with the pointer ending up opposite to the α and putting back the symbol(s) truncated in the above argument (from (41)) i.e. $\alpha_2\in\{\mathtt{a},\mathtt{c},\mathtt{d}\}$ and following the backward search algorithm from here to obtain all origins gives

$$\mathtt{4}\alpha_2\mathtt{c\underline{a}}\alpha \dashleftarrow \mathtt{3}\alpha_2\mathtt{\underline{a}a}\alpha\left\{\begin{array}{l} \stackrel{\alpha_2=\mathtt{a}}{\leftarrow}\left\{\begin{array}{l}\mathtt{5\underline{c}aa}\alpha\\ \mathtt{5\underline{e}aa}\alpha\end{array}\right. \\ \stackrel{\alpha_2=\mathtt{b}}{\leftarrow}\mathtt{3\underline{e}aa}\alpha \\ \stackrel{\alpha_2=\mathtt{c}}{\leftarrow}\mathtt{4\underline{c}aa}\alpha \\ \leftarrow\mathtt{1}\alpha_2\mathtt{a\underline{c}}\alpha\leftarrow\mathtt{2}\alpha_2\mathtt{\underline{d}c}\alpha\stackrel{\alpha_2=\mathtt{b}}{\leftarrow}\left\{\begin{array}{l}\mathtt{1\underline{d}dc}\alpha\\ \mathtt{1\underline{e}dc}\alpha\end{array}\right. \end{array}\right. . \tag{628}$$

This when restricted to $\alpha_2\in\{\mathtt{a},\mathtt{c},\mathtt{d}\}$ and summarised gives (629). All results derived from (1) similarly are listed now where I introduce the notation $\dashleftarrow$ to be like an incomplete $\leftarrow$ meaning that some of the results on the right may be omitted. Exactly how the selection has been made will be made clear whenever it is used.

$$\mathtt{4}\alpha_2\mathtt{c\underline{a}}\alpha \dashleftarrow \left\{\begin{array}{l} \stackrel{\alpha_2=\mathtt{a}}{\leftarrow}\left\{\begin{array}{l}\mathtt{5\underline{c}aa}\alpha\\ \mathtt{5\underline{e}aa}\alpha\end{array}\right. \\ \stackrel{\alpha_2=\mathtt{c}}{\leftarrow}\mathtt{4\underline{c}aa}\alpha \end{array}\right. \quad \text{for } \alpha_2\in\{\mathtt{a},\mathtt{c},\mathtt{d}\} \tag{629}$$

$$3\alpha_2\mathtt{a\underline{d}}\alpha \dashleftarrow \begin{cases} \overset{\alpha_2=\mathtt{a}}{\longleftarrow} \mathtt{4\underline{e}cd}\alpha \\ \overset{\alpha_2=\mathtt{a}}{\longleftarrow} \mathtt{4\underline{e}ed}\alpha \end{cases} \quad \text{for } \alpha_2 \in \{\mathtt{a},\mathtt{c},\mathtt{d}\} \tag{630}$$

$$2\alpha\underline{\beta}\gamma\mathtt{e} \dashleftarrow \begin{cases} \overset{\gamma=\mathtt{d}}{\longleftarrow} \begin{cases} \overset{\beta=\mathtt{a}}{\longleftarrow} \begin{cases} \overset{\alpha=\mathtt{b}}{\longleftarrow} \left\{ \mathtt{1\underline{d}dae} \quad \mathtt{1\underline{e}dae} \right\} \\ \mathtt{5}\alpha\mathtt{dc\underline{b}} \end{cases} \\ \overset{\beta=\mathtt{c}}{\longleftarrow} \begin{cases} \overset{\alpha=\mathtt{b}}{\longleftarrow} \left\{ \begin{matrix} \mathtt{1\underline{d}aae} & \mathtt{4\underline{b}cde} & \mathtt{1\underline{d}abe} \\ \mathtt{1\underline{e}aae} & \mathtt{3\underline{e}ade} & \mathtt{1\underline{e}abe} \end{matrix} \right\} \\ \longleftarrow \left\{ \mathtt{5}\alpha\mathtt{ac\underline{b}} \quad \mathtt{5}\alpha\mathtt{cd\underline{b}} \quad \mathtt{5}\alpha\mathtt{ed\underline{b}} \right\} \\ \overset{\alpha=\mathtt{a}}{\longleftarrow} \begin{cases} \mathtt{4\underline{e}cce} \quad \mathtt{5\underline{c}ade} \quad \mathtt{4\underline{e}ece} \\ \mathtt{5\underline{e}ade} \end{cases} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \left\{ \mathtt{3\underline{a}cde} \quad \mathtt{4\underline{c}ade} \right\} \end{cases} \\ \overset{\beta=\mathtt{d}}{\longleftarrow} \begin{cases} \overset{\alpha=\mathtt{b}}{\longleftarrow} \left\{ \mathtt{1\underline{d}cae} \quad \mathtt{4\underline{b}cce} \quad \mathtt{1\underline{e}cae} \quad \mathtt{4\underline{b}cbe} \right\} \\ \longleftarrow \left\{ \mathtt{5}\alpha\mathtt{cc\underline{b}} \quad \mathtt{5}\alpha\mathtt{cb\underline{a}} \quad \mathtt{5}\alpha\mathtt{cb\underline{b}} \right\} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \left\{ \mathtt{3\underline{a}cce} \quad \mathtt{3\underline{a}cbe} \right\} \end{cases} \end{cases} \\ \overset{\gamma=\mathtt{e},\beta=\mathtt{a}}{\longleftarrow} \begin{cases} \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4\underline{b}eae} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{3\underline{a}eae} \end{cases} \end{cases} \tag{631}$$

$$\mathtt{5aa\underline{d}}\alpha \dashleftarrow \emptyset \tag{632}$$

$$\mathtt{5ea\underline{d}}\alpha \dashleftarrow \emptyset \tag{633}$$

$$\mathtt{2}\alpha\mathtt{\underline{c}ee} \dashleftarrow \emptyset \tag{634}$$

$$\mathtt{3aa\underline{b}}\alpha \dashleftarrow \begin{cases} \mathtt{4\underline{e}cb}\alpha \\ \mathtt{4\underline{e}eb}\alpha \end{cases} \tag{635}$$

$$\mathtt{3ea\underline{b}}\alpha \dashleftarrow \emptyset \tag{636}$$

$$\mathtt{1aa\underline{b}}\alpha \dashleftarrow \emptyset \tag{637}$$

$$\mathtt{1ea\underline{b}}\alpha \dashleftarrow \emptyset \tag{638}$$

$$\mathtt{3ab\underline{d}}\alpha \dashleftarrow \begin{cases} \mathtt{5\underline{c}ed}\alpha \\ \mathtt{5\underline{e}ed}\alpha \end{cases} \tag{639}$$

$$\mathtt{3eb\underline{d}}\alpha \dashleftarrow \emptyset \tag{640}$$

$$3\alpha\underline{\mathtt{e}}\mathtt{ec} \dashleftarrow \begin{cases} 3\alpha\mathtt{eb}\underline{\mathtt{d}} \\ 4\alpha\mathtt{eb}\underline{\mathtt{d}} \end{cases} \tag{641}$$

$$3\alpha\underline{\mathtt{e}}\mathtt{ed} \dashleftarrow \emptyset \tag{642}$$

$$3\mathtt{bb}\underline{\mathtt{c}}\alpha \dashleftarrow \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} 3\mathtt{bee}\underline{\mathtt{c}} \\ \overset{\alpha=\mathtt{d}}{\leftarrow} 1\mathtt{bee}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{e}}{\leftarrow} 5\mathtt{bee}\underline{\mathtt{a}} \\ \leftarrow 3\underline{\mathtt{e}}\mathtt{ec}\alpha \end{cases} \tag{643}$$

$$3\mathtt{cb}\underline{\mathtt{c}}\alpha \dashleftarrow \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} 3\mathtt{cee}\underline{\mathtt{c}} \\ \overset{\alpha=\mathtt{d}}{\leftarrow} 1\mathtt{cee}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{e}}{\leftarrow} 5\mathtt{cee}\underline{\mathtt{a}} \\ \leftarrow 4\underline{\mathtt{c}}\mathtt{ec}\alpha \end{cases} \tag{644}$$

$$3\mathtt{eb}\underline{\mathtt{c}}\alpha \dashleftarrow \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} 3\mathtt{eee}\underline{\mathtt{c}} \\ \overset{\alpha=\mathtt{d}}{\leftarrow} 1\mathtt{eee}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{e}}{\leftarrow} 5\mathtt{eee}\underline{\mathtt{a}} \end{cases} \tag{645}$$

$$4\alpha_2\mathtt{b}\underline{\mathtt{a}}\alpha \dashleftarrow \begin{cases} \overset{\alpha_2=\mathtt{b}}{\leftarrow} 4\underline{\mathtt{b}}\mathtt{ba}\alpha \\ \overset{\alpha_2=\mathtt{c}}{\leftarrow} 3\underline{\mathtt{a}}\mathtt{ba}\alpha \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \left\{ \; 3\alpha_2\mathtt{bd}\underline{\mathtt{d}} \quad 4\alpha_2\mathtt{bd}\underline{\mathtt{d}} \; \right\} \end{cases} \quad \text{for } \alpha_2 \in \{\mathtt{b},\mathtt{c},\mathtt{e}\} \tag{646}$$

$$2\alpha_2\mathtt{b}\underline{\mathtt{e}}\alpha \overset{\alpha_2=\mathtt{c}}{\dashleftarrow} \begin{cases} 2\underline{\mathtt{a}}\mathtt{de}\alpha \\ 2\underline{\mathtt{a}}\mathtt{ee}\alpha \end{cases} \quad \text{for } \alpha_2 \in \{\mathtt{b},\mathtt{c},\mathtt{e}\} \tag{647}$$

$$4\mathtt{eb}\underline{\mathtt{d}}\alpha \dashleftarrow \emptyset \tag{648}$$

$$4\mathtt{ab}\underline{\mathtt{d}}\alpha \dashleftarrow \emptyset \tag{649}$$

$$2\alpha\underline{\mathtt{d}}\mathtt{ee} \dashleftarrow \emptyset \tag{650}$$

$$3\alpha\underline{\mathtt{e}}\mathtt{aa} \dashleftarrow \emptyset \tag{651}$$

$$3\alpha\underline{\mathtt{a}}\mathtt{b}\alpha_2 \dashleftarrow \begin{cases} \overset{\alpha_2=\mathtt{a}}{\leftarrow} 5\alpha\mathtt{aa}\underline{\mathtt{d}} \\ \overset{\alpha_2=\mathtt{d}}{\leftarrow} 1\alpha\mathtt{aa}\underline{\mathtt{b}} \end{cases} \tag{652}$$

$$4\alpha\underline{\mathtt{c}}\mathtt{aa} \dashleftarrow \emptyset \tag{653}$$

$$1\alpha_2\mathtt{d}\underline{\mathtt{c}}\alpha \dashleftarrow \emptyset \text{ for } \alpha_2 \in \{\mathtt{c},\mathtt{e}\} \tag{654}$$

$$1\alpha_2\texttt{d}\underline{\texttt{b}}\alpha \dashleftarrow \emptyset \text{ for } \alpha_2 \in \{\texttt{c},\texttt{e}\} \tag{655}$$

$$3\alpha\underline{\texttt{a}}\texttt{ed} \dashleftarrow \left\{ \begin{array}{l} \overset{\alpha=\texttt{b}}{\longleftarrow} 4\underline{\texttt{b}}\texttt{ebd} \\ \overset{\alpha=\texttt{c}}{\longleftarrow} 3\underline{\texttt{a}}\texttt{ebd} \end{array} \right. \tag{656}$$

$$4\alpha\underline{\texttt{e}}\texttt{cd} \dashleftarrow 1\alpha\texttt{eb}\underline{\texttt{a}} \tag{657}$$

$$4\alpha\underline{\texttt{e}}\texttt{cb} \dashleftarrow 4\alpha\texttt{eb}\underline{\texttt{a}} \tag{658}$$

Only the results in the set (629)-(657) where the pointer in the origin (i.e. the RHS) is at the same end as the α will give rise to any members of IRR(4) because the others do not establish the reachability of the LHS. The result (629) does not satisfy this condition. An example that does satisfy it is from (631)

$$2\alpha\underline{\texttt{d}}\texttt{de} \overset{\alpha=\texttt{b}}{\dashleftarrow} \left\{ \begin{array}{l} 1\underline{\texttt{d}}\texttt{cae} \\ 1\underline{\texttt{e}}\texttt{cae} \\ 4\underline{\texttt{b}}\texttt{cce} \\ 4\underline{\texttt{b}}\texttt{cbe} \end{array} \right. \tag{659}$$

The context in which this arose i.e. the member of IRR(3) concerned (45).1 has type RL so it is potentially extendable to members of IRR(4) and these results can be combined to give the following result in IRR(4) where the LHS is $\texttt{2bab}\underline{\texttt{e}}$

$$\left. \begin{array}{l} 1\underline{\texttt{d}}\texttt{cae} \\ 1\underline{\texttt{e}}\texttt{cae} \\ 4\underline{\texttt{b}}\texttt{cce} \\ 4\underline{\texttt{b}}\texttt{cbe} \end{array} \right\} \rightarrow 2\texttt{b}\underline{\texttt{d}}\texttt{de} \rightarrow 2\texttt{bab}\underline{\texttt{e}} \rightarrow 3\underline{\texttt{b}}\texttt{bcc} \rightarrow 4\_\texttt{cbcc} \tag{660}$$

which gives the IRR

$$\left. \begin{array}{l} 1\underline{\texttt{d}}\texttt{cae} \\ 1\underline{\texttt{e}}\texttt{cae} \\ 4\underline{\texttt{b}}\texttt{cce} \\ 4\underline{\texttt{b}}\texttt{cbe} \end{array} \right\} \rightarrow 2\texttt{bab}\underline{\texttt{e}} \rightarrow 4\_\texttt{cbcc} \tag{661}$$

Doing this for the remaining results in the list (629)-(657) gives the following results for IRR(4) in triplet form

$$\left. \begin{array}{l} 1\underline{\texttt{d}}\texttt{cae} \\ 1\underline{\texttt{e}}\texttt{cae} \\ 4\underline{\texttt{b}}\texttt{cce} \\ 4\underline{\texttt{b}}\texttt{cbe} \end{array} \right\} \rightarrow 2\texttt{bab}\underline{\texttt{e}} \rightarrow 4\_\texttt{cbcc} \tag{662}$$

$$\left. \begin{array}{lll} 1\underline{\texttt{d}}\texttt{aae} & 4\underline{\texttt{b}}\texttt{cde} & 1\underline{\texttt{d}}\texttt{abe} \\ 1\underline{\texttt{e}}\texttt{aae} & 3\underline{\texttt{e}}\texttt{ade} & 1\underline{\texttt{e}}\texttt{abe} \end{array} \right\} \rightarrow 2\texttt{bdb}\underline{\texttt{e}} \rightarrow 3\_\texttt{eccc} \tag{663}$$

$$\left.\begin{array}{l}\texttt{4\underline{e}cce}\\ \texttt{5\underline{c}ade}\\ \texttt{4\underline{e}ece}\\ \texttt{5\underline{e}ade}\end{array}\right\} \to \texttt{2adb\underline{e}} \to \texttt{2\_eccc} \tag{664}$$

$$\left.\begin{array}{l}\texttt{3\underline{a}cde}\\ \texttt{4\underline{c}ade}\end{array}\right\} \to \texttt{2cdb\underline{e}} \to \texttt{5\_ceca} \tag{665}$$

$$\left.\begin{array}{l}\texttt{3\underline{a}cce}\\ \texttt{3\underline{a}cbe}\end{array}\right\} \to \texttt{2cab\underline{e}} \to \texttt{2\_abcc} \tag{666}$$

$$\texttt{3bee\underline{c}} \to \texttt{4\underline{b}caa} \to \texttt{3babc\_} \tag{667}$$

$$\texttt{1bee\underline{a}} \to \texttt{4\underline{b}cad} \to \texttt{2bcdb\_} \tag{668}$$

$$\texttt{5bee\underline{a}} \to \texttt{4\underline{b}cae} \to \texttt{5bcca\_} \tag{669}$$

$$\texttt{3eee\underline{c}} \to \texttt{4\underline{e}caa} \to \texttt{3adbc\_} \tag{670}$$

$$\texttt{1cee\underline{a}} \to \texttt{4\underline{c}cad} \to \texttt{2abcb\_} \tag{671}$$

$$\texttt{5cee\underline{a}} \to \texttt{4\underline{c}cae} \to \texttt{2abcb\_} \tag{672}$$

$$\texttt{3cee\underline{c}} \to \texttt{4\underline{c}caa} \to \texttt{2abdb\_} \tag{673}$$

$$\texttt{1eee\underline{a}} \to \texttt{4\underline{e}cad} \to \texttt{2aadb\_} \tag{674}$$

$$\texttt{5eee\underline{a}} \to \texttt{4\underline{e}cae} \to \texttt{5aaca\_} \tag{675}$$

$$\left.\begin{array}{l}\texttt{3bbd\underline{d}}\\ \texttt{4bbd\underline{d}}\end{array}\right\} \to \texttt{4\underline{b}cbc} \to \texttt{1babd\_} \tag{676}$$

$$\left.\begin{array}{l}\texttt{3cbd\underline{d}}\\ \texttt{4cbd\underline{d}}\end{array}\right\} \to \texttt{4\underline{c}cbc} \to \texttt{2abab\_} \tag{677}$$

$$\left.\begin{array}{l}\texttt{3ebd\underline{d}}\\ \texttt{4ebd\underline{d}}\end{array}\right\} \to \texttt{4\underline{e}cbc} \to \texttt{2cbab\_} \tag{678}$$

$$\texttt{4\underline{b}ebd} \to \texttt{5bca\underline{d}} \to \texttt{4\_caba} \tag{679}$$

$$\mathtt{3\underline{a}ebd} \rightarrow \mathtt{5cca\underline{d}} \rightarrow \mathtt{4abcc\_} \tag{680}$$

This argument shows that there are other ways to generate IRR's from existing ones that are not contained in the Tables 2 and 22 by making use of the cases explicitly excluded by the Search Condition.

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A more general approach to this is to search for a given outline in the column headed IRR($\mathtt{n}+\mathtt{1}$) to find the IRR($\mathtt{n}$) and α from which it is derived, then go forwards again with the derivation but this time do not truncate the extra symbol. This results in an IRR truncated triplet of length $\mathtt{k} = \mathtt{3}$. Then this can be used as a starting point for a derivation analogous to those summarised in Table 13 but giving results having length $\mathtt{k} = \mathtt{3}$. This will now be carried out using Theorem 9.1 and Table 12.

The starting points for these derivations are the entries in Tables 2 and 22, from the abbreviated IRR triplet (A) of length $\mathtt{k}$ of an IRR of length $\mathtt{n}$, under the heading IRR($\mathtt{n}$) to where the derivation of the new origins go unexpectedly to the opposite end of the string from the arbitrary symbol α. This results in an abbreviated non-IRR triplet (B) of length $\mathtt{k}+\mathtt{1}$ in the last column that does not represent IRR's because the pointer is at the wrong end in the origin, so not demonstrating the reachability of the LHS. These cases could represent incomplete derivations of new abbreviated IRR triplets where the pointer could move back to where the α is if there were more symbols included. An extra symbol in A is found by searching in Tables 2 or 22 (the one which contains $\mathtt{A}$) for all possible entries $\mathtt{C}$ under IRR($\mathtt{n}$) that lead to $\mathtt{A}$ in the last column, and for each such $\mathtt{C}$, rederive $\mathtt{A}$ (now called $\mathtt{A}'$) without deleting the extra symbol thus $\mathtt{A}'$ has length $\mathtt{k}+\mathtt{1}$. Then for each $\mathtt{A}'$, add the arbitrary symbol α as in the general procedure in section 2 and carry out the backward search to find all the new origins for each value of α. Also the new RHS's must be found from the RHS of $\mathtt{A}'$ for each α. Then for each α all the derived IRR triplets having length $\mathtt{k}+\mathtt{1}$ symbols specified are listed. If there are any cases where the backward search algorithm leads to the opposite end of the string from the α, these are included in full (length $\mathtt{k}+\mathtt{2}$)

Are the results for $\mathtt{n} = \mathtt{4}$ included, i.e those derived by keeping an extra symbol at one point?

For example, the abbreviated IRR $\mathtt{A} = \mathtt{2\underline{a}d}\ldots \rightarrow\rightarrow \mathtt{2\_ae}\ldots$ leads to the abbreviated non-IRR $\mathtt{B} = \mathtt{5}\alpha\mathtt{a\underline{a}}\ldots \rightarrow\rightarrow \mathtt{RHS}$ where the RHS is dependent on α and is not specified (can be worked out easily).

Table 2 shows that $\mathtt{C} = \mathtt{1\underline{e}c}\ldots \rightarrow \{\mathtt{1},\mathtt{2}\}\mathtt{bd}\ldots \rightarrow \mathtt{3\_ec}\ldots$ leads to $\mathtt{A}$. Then rederive $\mathtt{A}$ from $\mathtt{C}$, which starts by finding all origins of $\mathtt{1}\alpha\mathtt{\underline{e}c}\ldots$ i.e. $\mathtt{2\underline{d}ec}\ldots$, $\mathtt{2\underline{a}ec}\ldots$, and $\mathtt{2\underline{c}ec}\ldots$ for $\alpha = \mathtt{a},\mathtt{c},\mathtt{d}$ respectively. The result matches the origin of $\mathtt{A}$ only if $\alpha = \mathtt{c}$ therefore $\mathtt{A}' = \mathtt{2\underline{a}ec} \rightarrow \{\mathtt{1},\mathtt{2}\}\mathtt{cbd}\ldots \rightarrow \mathtt{2\_aec}\ldots$. Then apply $\mathtt{F}$ again, this time ignoring the result leading to line 12 of Table 2 because it has already been found, starting with the origins of $\mathtt{2}\alpha\mathtt{\underline{a}ec}\ldots$ given by

$$\mathtt{2\alpha\underline{a}ec}\ldots \leftarrow \mathtt{5\alpha a\underline{a}c}\ldots \leftarrow \begin{cases} \mathtt{3\alpha aa\underline{d}}\ldots \\ \mathtt{4\alpha aa\underline{d}}\ldots \\ \mathtt{4\alpha\underline{e}ac}\ldots \begin{cases} \leftarrow \mathtt{5\alpha e\underline{d}c}\ldots \leftarrow \begin{cases} \mathtt{3\alpha ed\underline{d}}\ldots \\ \mathtt{4\alpha ed\underline{d}}\ldots \end{cases} \\ \overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{4\underline{b}eac}\ldots \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{3\underline{a}eac}\ldots \end{cases} \end{cases} . \tag{681}$$

Finally, after calculating the new RHS's thus $\mathtt{2\underline{b}aec}\ldots \rightarrow \mathtt{4\_caec}\ldots$ and $\mathtt{2\underline{c}aec}\ldots \rightarrow \mathtt{3\_bccc}\ldots$ and truncating to $\mathtt{k} = 3$ symbols and filling in the LHS's gives the second major row of Table 20.

Table 20: A few outline IRRs of length $\mathtt{n}$ and their derived outline IRR of length $\mathtt{n}+\mathtt{1}$, with the unknown symbols on the right, for $\mathtt{k} = 3$ symbols

row# in	IRR(n)			IRR(n + 1)		
table 2	Origin	LHS	RHS	Origin	LHS	RHS
9	$\mathtt{2\underline{a}dc}\ldots$	{1,2}cbd...	2_aec...	$\mathtt{1\underline{d}ad}\ldots$	xbcb...	4_cae...
				$\mathtt{1\underline{e}ad}\ldots$	xbcb...	4_cae...
				$\mathtt{1\underline{d}da}\ldots$	xbcb...	4_cae...
				$\mathtt{1\underline{e}da}\ldots$	xbcb...	4_cae...
				$\mathtt{2\alpha dc\underline{e}}\ldots$	$\mathtt{x\alpha cbd}\ldots$	RHS
12	$\mathtt{2\underline{a}ec}\ldots$	{1,2}cbd...	2_aec...	$\mathtt{4\underline{b}ea}\ldots$	xbcb...	4_cae...
				$\mathtt{3\underline{a}ea}\ldots$	xccb...	3_bcc...
				$\mathtt{3\alpha aa\underline{d}}\ldots$	$\mathtt{x\alpha cbd}\ldots$	RHS
				$\mathtt{4\alpha aa\underline{d}}\ldots$	$\mathtt{x\alpha cbd}\ldots$	RHS
				$\mathtt{3\alpha ed\underline{d}}\ldots$	$\mathtt{x\alpha cbd}\ldots$	RHS
				$\mathtt{4\alpha ed\underline{d}}\ldots$	$\mathtt{x\alpha cbd}\ldots$	RHS
15	$\mathtt{2\underline{c}da}\ldots$	{1,2}dbc...	5_cca...	$\mathtt{4\underline{e}cc}\ldots$	xadb...	2_ecc...
				$\mathtt{4\underline{e}ec}\ldots$	xadb...	2_ecc...
				$\mathtt{5\underline{c}ad}\ldots$	xadb...	2_ecc...
				$\mathtt{5\underline{e}ad}\ldots$	xadb...	2_ecc...
				$\mathtt{1\underline{d}cd}\ldots$	xbdb...	3_ecc...
				$\mathtt{1\underline{e}cd}\ldots$	xbdb...	3_ecc...
				$\mathtt{1\underline{d}aa}\ldots$	xbdb...	3_ecc...
				$\mathtt{1\underline{e}aa}\ldots$	xbdb...	3_ecc...
				$\mathtt{4\underline{b}cd}\ldots$	xbdb...	3_ecc...
				$\mathtt{3\underline{e}ad}\ldots$	xbdb...	3_ecc...
				$\mathtt{1\underline{d}ab}\ldots$	xbdb...	3_ecc...
				$\mathtt{1\underline{e}ab}\ldots$	xbdb...	3_ecc...
				$\mathtt{3\underline{a}cd}\ldots$	xcdb...	1dbd...
				$\mathtt{4\underline{c}ad}\ldots$	xcdb...	1dbd...
				$\mathtt{1\alpha ac\underline{c}}\ldots$	$\mathtt{x\alpha dbc}\ldots$	RHS
				$\mathtt{1\alpha cd\underline{c}}\ldots$	$\mathtt{x\alpha dbc}\ldots$	RHS
				$\mathtt{5\alpha cd\underline{d}}\ldots$	$\mathtt{x\alpha dbc}\ldots$	RHS
				$\mathtt{1\alpha ed\underline{c}}\ldots$	$\mathtt{x\alpha dbc}\ldots$	RHS
				$\mathtt{5\alpha ed\underline{d}}\ldots$	$\mathtt{x\alpha dbc}\ldots$	RHS
15	$\mathtt{2\underline{c}dd}\ldots$	1dba...	5_cca...	$\mathtt{5\underline{c}ad}\ldots$	1adb...	2_ecc...
				$\mathtt{4\underline{e}cc}\ldots$	1adb...	2_ecc...
				$\mathtt{5\underline{e}ad}\ldots$	1adb...	2_ecc...
				$\mathtt{4\underline{e}ec}\ldots$	1adb...	2_ecc...
				$\mathtt{1\underline{d}cd}\ldots$	1bdb...	3_ecc...
				$\mathtt{1\underline{e}cd}\ldots$	1bdb...	3_ecc...
				$\mathtt{1\underline{d}ab}\ldots$	1bdb...	3_ecc...
				$\mathtt{1\underline{e}ab}\ldots$	1bdb...	3_ecc...

		1$\underline{\text{d}}$aa...	1bdb...	3_ecc...
		1$\underline{\text{e}}$aa...	1bdb...	3_ecc...
		4$\underline{\text{b}}$cd...	1bdb...	3_ecc...
		3$\underline{\text{e}}$ad...	1bdb...	3_ecc...
		3$\underline{\text{a}}$cd...	1cdb...	2dba...
		4$\underline{\text{c}}$ad...	1cdb...	2dba...
		1αcd$\underline{\text{b}}$...	1αdba...	RHS
		1αed$\underline{\text{b}}$...	1αdba...	RHS
15	2$\underline{\text{c}}$dd... {1,2}dba... 5_ccb...	5$\underline{\text{c}}$ad...	xadb...	2_ecc...
		4$\underline{\text{e}}$cc...	xadb...	2_ecc...
		5$\underline{\text{e}}$ad...	xadb...	2_ecc...
		4$\underline{\text{e}}$ec...	xadb...	2_ecc...
		1$\underline{\text{d}}$cd...	xbdb...	3_ecc...
		1$\underline{\text{e}}$cd...	xbdb...	3_ecc...
		1$\underline{\text{d}}$ab...	xbdb...	3_ecc...
		1$\underline{\text{e}}$ab...	xbdb...	3_ecc...
		1$\underline{\text{d}}$aa...	xbdb...	3_ecc...
		1$\underline{\text{e}}$aa...	xbdb...	3_ecc...
		4$\underline{\text{b}}$cd...	xbdb...	3_ecc...
		3$\underline{\text{e}}$ad...	xbdb...	3_ecc...
		3$\underline{\text{a}}$cd...	xcdb...	5_cec...
		4$\underline{\text{c}}$ad...	xcdb...	5_cec...
		1αcd$\underline{\text{b}}$...	xαdba...	RHS
		1αed$\underline{\text{b}}$...	xαdba...	RHS
16	2$\underline{\text{c}}$dc... {1,2}dbd... 5_cec...	4$\underline{\text{e}}$cc...	xadb...	2_ece...
		5$\underline{\text{e}}$ad...	xadb...	2_ece...
		5$\underline{\text{c}}$ad...	xadb...	2_ece...
		4$\underline{\text{e}}$ec...	xadb...	2_ece...
		1$\underline{\text{d}}$cd...	xbdb...	3_ece...
		1$\underline{\text{e}}$cd...	xbdb...	3_ece...
		1$\underline{\text{d}}$aa...	xbdb...	3_ece...
		1$\underline{\text{e}}$aa...	xbdb...	3_ece...
		1$\underline{\text{d}}$ab...	xbdb...	3_ece...
		1$\underline{\text{e}}$ab...	xbdb...	3_ece...
		4$\underline{\text{b}}$cd...	xbdb...	3_ece...
		3$\underline{\text{e}}$ad...	xbdb...	3_ece...
		3$\underline{\text{a}}$cd...	xcdb...	5_ccc...
		4$\underline{\text{c}}$ad...	xcdb...	5_ccc...
		2αac$\underline{\text{e}}$...	xαdbd...	RHS
		2αcd$\underline{\text{e}}$...	xαdbd...	RHS
		2αcd$\underline{\text{b}}$...	xαdbd...	RHS
		3αcd$\underline{\text{b}}$...	xαdbd...	RHS
		2αed$\underline{\text{e}}$...	xαdbd...	RHS
		2αed$\underline{\text{b}}$...	xαdbd...	RHS
		3αed$\underline{\text{b}}$...	xαdbd...	RHS

This table is incomplete because it would take up too much space if written out in full like this.

If these results are put into groups where just the symbol furthest from the α (i.e. on the right here) is different in the origins of the IRR(n) outlines, the sets of origins of the corresponding IRR(n + 1) outlines for each value of α are the same in each member of the group. This is because in the backward search, the pointer reaches the symbols that differ at the point where the backward search algorithm terminates, and the symbols that differ are truncated off the result. For example comparing the backward search algorithm applied to 2$\alpha\underline{\text{c}}$dd... and 2$\alpha\underline{\text{c}}$dc..., the results will be identical, apart from the change in the last symbol, up to the point where the pointer first reaches the right hand end of the given symbols. The last backward step will be for $\_\mathtt{d}\ldots$ in the first case and $\_\mathtt{c}\ldots$ in the second thus leaving identical results apart from the last symbol that is then truncated off.

As a result of this, the essential content of the complete version of Table 20 (for abbreviated length $\mathtt{k} = \mathtt{3}$) can be expressed much more compactly by giving just

1. the list of outline IRR($\mathtt{n}$) triplets ($\mathtt{A}'$)
2. for each symbol α and distinct origin of the IRR($\mathtt{n}$) outlines abbreviated further (now having length $\mathtt{k}-\mathtt{1}$) by deleting the symbol opposite the α (β), the set of corresponding origins of IRR($\mathtt{n}+\mathtt{1}$) outlines (length $\mathtt{k}$)
3. For each distinct abbreviated origin of the IRR($\mathtt{n}$) outlines and β, which RHS's of the IRR($\mathtt{n}$) outlines applies.
4. for each symbol α and distinct RHS of IRR($\mathtt{n}$) outlines of length $\mathtt{k}$, the RHS of IRR($\mathtt{n}+\mathtt{1}$) outlines of length $\mathtt{k}$
5. for each distinct origin of IRR($\mathtt{n}$) outlines length $\mathtt{k}$, the set of all new non-IRR triplet origins of length $\mathtt{k}+\mathtt{1}$.

This assumes that the user can fill in the LHS's of IRR($\mathtt{n}+\mathtt{1}$) themselves. Table irr origins contain the information in (2) for Tables 2 and 22.

Table 21 contains the initial list $\mathtt{A}'$ for $\mathtt{k} = \mathtt{3}$ which must be extended to closure. Tables 16 17 and 18 contain the information in (2) (3) and (4) respectively for $\mathtt{k} = \mathtt{3}$ after closure has been applied. This would be done by including in the IRR($\mathtt{n}$) column of Table 20 those IRR($\mathtt{n}+\mathtt{1}$) that are not already there, unless the IRR($\mathtt{n}+\mathtt{1}$) matches an IRR($\mathtt{n}$) in Table irr origins and completing the row of the Table 20 etc. until there are no further changes. The point of the last exception is to ensure that any IRR triplet represented in the completion of Table 20 can have $\mathtt{F}$ applied in accordance with either Table irr origins or the completion of Table 20 i.e. Tables 16 17 and 18.

How do these 3 tables represent the essential content of the completion of Table 20? Table 16 is the most complicated and is divided into major rows by horizontal lines. Each major row starts from a set of origin CS's given by the top left CS with the values of β indicated in the second column. Applying the backward search algorithm starting from the origin CS's with the arbitrary symbol α added adjacent to the pointer gives a search tree ending with CS's having the pointer at either end of the string, unless condition 3 is reached when no further computation can be done. If the pointer ends in condition 1 with the pointer where α was, the results (the new origins) truncated to the same number of symbols ($\mathtt{k} = \mathtt{3}$) are listed in Table 16 under the appropriate value of α. If the computation ends with condition 2 i.e. the pointer ends up at the opposite end from the α, the result appears in Table 18. The RHS's corresponding to the original CS's are listed below them. Note that there is not a one-to-one correspondence between the different origins and the RHS's, and not every combination of β and RHS is necessarily possible together. This is detail not contained in Table 16 and is relatively unimportant because every origin CS and symbol α gives rise to the same set of new origins for IRR($\mathtt{n}+\mathtt{1}$) which are those listed. The new RHS's of IRR($\mathtt{n}+\mathtt{1}$) depend only on the the RHS of IRR($\mathtt{n}$) and are therefore listed separately in Table 17. Similarly for the cases that are the new exceptions i.e. where the pointer goes away from the α in the backward search algorithm, then new origin depends only on the origin of IRR($\mathtt{n}$) and are listed in Table 18.

Table 21: Table of extended IRR triplets ($\mathtt{A}$) and ($\mathtt{A}'$)

	triplet IRR(n) with k = 2			extended triplet with k = 3		
9	2ad...	{1,2}cb...	2_ae...	2adc...	xcbd...	2_aec...
12	2ae...	{1,2}cb...	2_ae...	2aec...	xcbd...	2_aec...
15	2cd...	{1,2}db...	5_cc...	2cda...	{1,2}dbc...	5_cca...
				2cdd...	1dba...	5_cca...
				2cdd...	{1,2}dba...	5_ccb...
16	2cd...	{1,2}db...	5_ce...	2cdc...	{1,2}dbd...	5_cec...
17	2ce...	{1,2}db...	5_cc...	2cea...	{1,2}dbc...	5_cca...
				2ced...	1dba...	5_cca...
				2ced...	{1,2}dba...	5_ccb...
18	2ce...	{1,2}db...	5_ce...	2cec...	{1,2}dbd...	5_cec...
19	2dd...	{1,2}ab...	3_bc...	2ddd...	1aba...	3_bca...
				2ddd...	{1,2}aba...	3_bcb...
				2dda...	{1,2}abc...	3_bca...
21	2de...	{1,2}ab...	3_bc...	2dea...	{1,2}abc...	3_bca...
				2ded...	1aba...	3_bca...
				2ded...	{1,2}aba...	3_bcb...
23	3ab...	4cb...	2_ae...			
24	3ab...	4cb...	3_bc...	3abb...	4cbb...	3_bcc...
				3abe...	4cba...	3_bcc...
27	3ac...	{3,4}cc...	2_ab...	3ace...	3ccb...	2_abc...
				3aca...	4ccc...	2_abc...
28	3ac...	3cc...	3_bc...	3ace...	3ccb...	3_bcc...
30	3ae...	5ca...	2_ab...			
32	3ea...	{3,4,5}bc...	4_ca...	3eab...	4bcb...	4_cae...
				3eac...	{3,4}bcc...	4_cab...
				3eae...	5bca...	4_cab...
33	3ea...	{3,4}bc...	4_cb...	3eab...	4bcb...	4_cbc...
				3eac...	3bcc...	4_cbc...
34	3ee...	3bb...	4_ce...			
42	4bc...	{3,4}bc...	4_ca...	4bca...	{3,4}bcc...	4_cab...
				4bce...	3bcb...	4_cae...
43	4bc...	{3,4}bc...	4_cb...	4bca...	4bcc...	4_cbc...
				4bce...	3bcb...	4_cbc...
46	4ca...	{3,4}cc...	2_ab...	4cab...	4ccb...	2_abc...
				4cac...	3ccc...	2_abc...
47	4ca...	4cc...	3_bc...	4cab...	4ccb...	3_bcc...
53	4ec...	4aa...	2_ec...	4eca...	4aac...	2_ecc...
9	1...db	{3,4,5}...cd	1...ba_	1...ddb	{3,4}...acd	1...aba_
10	1...db	{3,4}...cd	1...bd_	1...bdb	{3,4}...ecd	1...cbd_
11	1...db	{2,3,4,5}...cd	2...cb_	1...adb	{3,4}...ecd	2...bcb_
				1...bdb	{2,3,4,5}...ecd	2...bcb_

12	1...d$\underline{b}$	{2,3,4,5}...cd	2...db_	1...ad$\underline{b}$	{2,3,4,5}...ecd	2...bdb_
				1...bd$\underline{b}$	{2,3,4,5}...ecd	2...bdb_
				1...dd$\underline{b}$	{2,3,4,5}...acd	2...bdb_
				1...dd$\underline{b}$	{2,3,4,5}...acd	2...cdb_
				1...bd$\underline{b}$	3...ecd	2...cdb_
17	1...c$\underline{c}$	{2,3,4,5}...aa	1...bc_	1...bc$\underline{c}$	{2,3,4,5}...caa	1...abc_
				1...ec$\underline{c}$	{2,3,4,5}...caa	1...abc_
18	1...c$\underline{c}$	{2,3,4,5}...aa	2...db_	1...bc$\underline{c}$	{2,3,4,5}...caa	2...bdb_
				1...ec$\underline{c}$	{2,3,4,5}...caa	2...bdb_
19	1...c$\underline{c}$	{3,4}...aa	3...bc_	1...bc$\underline{c}$	4...caa	3...dbc_
				1...bc$\underline{c}$	{3,4}...caa	3...abc_
				1...ec$\underline{c}$	{3,4}...caa	3...abc_
20	1...d$\underline{c}$	{3,4,5}...ca	1...bc_	1...dd$\underline{c}$	{3,4}...aca	1...abc_
21	1...d$\underline{c}$	{2,3,4,5}...ca	2...ab_	1...ad$\underline{c}$	{2,3,4,5}...eca	2...bab_
				1...bd$\underline{c}$	{2,3,4,5}...eca	2...bab_
				1...dd$\underline{c}$	{2,3,4,5}...aca	2...bab_
22	1...d$\underline{c}$	{2,3,4,5}...ca	2...db_	1...ad$\underline{c}$	{3,4}...eca	2...bdb_
				1...bd$\underline{c}$	{2,3,4,5}...eca	2...bdb_
23	1...d$\underline{c}$	3...ca	3...bc_	1...bd$\underline{c}$	3...eca	3...abc_
24	1...d$\underline{c}$	{3,4}...ca	4...ac_	1...bd$\underline{c}$	{3,4}...eca	4...aac_
25	1...d$\underline{c}$	{2,3,4,5}...ca	4...cc_	1...dd$\underline{c}$	{2,3,4,5}...aca	4...ccc_
36	2...b$\underline{e}$	{2,3,4,5}...cc	1...bd_	2...db$\underline{e}$	{2,3,4,5}...ccc	1...abd_
				2...db$\underline{e}$	{3,4}...ccc	1...dbd_
				2...ab$\underline{e}$	{2,3,4,5}...bcc	1...abd_
37	2...b$\underline{e}$	{3,4}...cc	2...ab_	2...ab$\underline{e}$	{3,4}...bcc	2...bab_
				2...db$\underline{e}$	3...ccc	2...cab_
38	2...b$\underline{e}$	4...cc	2...db_			
48	3...a$\underline{b}$	{2,3,4,5}...bc	1...bd_	3...ba$\underline{b}$	{3,4}...cbc	1...abd_
				3...ca$\underline{b}$	{2,3,4,5}...abc	1...dbd_
				3...da$\underline{b}$	{2,3,4,5}...cbc	1...abd_
				3...da$\underline{b}$	{2,3,4,5}...cbc	1...dbd_
49	3...a$\underline{b}$	{2,3,4,5}...bc	2...ab_	3...ba$\underline{b}$	{2,3,4,5}...cbc	2...bab_
				3...ca$\underline{b}$	{2,3,4,5}...cbc	2...bab_
				3...da$\underline{b}$	{3,4,5}...cbc	2...bab_
50	3...a$\underline{b}$	{3,4}...bc	3...bc_	3...ca$\underline{b}$	{3,4}...abc	3...cbc_
				3...da$\underline{b}$	3...cbc	3...cbc_
57	3...b$\underline{c}$	{2,3,4,5}...ca	1...bc_	3...ab$\underline{c}$	{2,3,4,5}...bca	1...abc_
				3...db$\underline{c}$	{2,3,4,5}...cca	1...abc_
				3...db$\underline{c}$	{3,4}...cca	1...dbc_
58	3...b$\underline{c}$	{2,3,4,5}...ca	2...ab_	3...ab$\underline{c}$	{2,3,4,5}...bca	2...bab_
				3...db$\underline{c}$	{3,4,5}...cca	2...bab_
59	3...b$\underline{c}$	4...ca	4...ac_			
60	3...b$\underline{c}$	{3,4}...ca	4...cc_	3...ab$\underline{c}$	{3,4}...bca	4...bcc_
				3...db$\underline{c}$	3...cca	4...ccc_
65	3...a$\underline{d}$	{3,4}...ec	1...bc_	3...ba$\underline{d}$	{3,4}...cec	1...abc_
				3...ea$\underline{d}$	{3,4}...cec	1...abc_
66	3...a$\underline{d}$	{2,3,4,5}...ec	1...bd_	3...ba$\underline{d}$	{2,3,4,5}...cec	1...dbd_
				3...ea$\underline{d}$	{2,3,4,5}...cec	1...dbd_
68	3...b$\underline{d}$	{2,3,4,5}...ec	1...bc_	3...bb$\underline{d}$	{3,4}...cec	1...abc_
				3...db$\underline{d}$	{2,3,4,5}...cec	1...abc_
69	3...b$\underline{d}$	{2,3,4,5}...ec	1...bd_	3...bb$\underline{d}$	{2,3,4,5}...cec	1...dbd_
				3...db$\underline{d}$	{2,3,4,5}...cec	1...cbd_
				3...db$\underline{d}$	{2,3,4,5}...cec	1...dbd_
				3...cb$\underline{d}$	{2,3,4,5}...aec	1...cbd_
70	3...b$\underline{d}$	{3,4}...ec	3...aa_	3...cb$\underline{d}$	{3,4}...aec	3...caa_
				3...db$\underline{d}$	3...cec	3...caa_

71	3...b$\underline{\mathtt{d}}$	3...ec	4...cc_			
77	4...b$\underline{\mathtt{a}}$	{3,4}...cb	2...ab_	4...ab$\underline{\mathtt{a}}$	{3,4}...bcb	2...bab_
				4...db$\underline{\mathtt{a}}$	3...ccb	2...cab_
78	4...b$\underline{\mathtt{a}}$	{2,3,4,5}...cb	3...bc_	4...db$\underline{\mathtt{a}}$	{2,3,4,5}...ccb	3...abc_
				4...db$\underline{\mathtt{a}}$	{3,4}...ccb	3...dbc_
				4...ab$\underline{\mathtt{a}}$	{2,3,4,5}...bcb	3...abc_
79	4...c$\underline{\mathtt{a}}$	{2,3,4,5}...ab	2...db_	4...bc$\underline{\mathtt{a}}$	{2,3,4,5}...cab	2...bdb_
				4...ec$\underline{\mathtt{a}}$	{2,3,4,5}...cab	2...bdb_
80	4...c$\underline{\mathtt{a}}$	{2,3,4,5}...ab	3...bc_	4...bc$\underline{\mathtt{a}}$	{2,3,4,5}...cab	3...abc_
				4...ec$\underline{\mathtt{a}}$	{2,3,4,5}...cab	3...abc_
81	4...c$\underline{\mathtt{a}}$	{3,4}...ab	4...cb_	4...bc$\underline{\mathtt{a}}$	4...cab	4...acb_
				4...bc$\underline{\mathtt{a}}$	{3,4}...cab	4...ccb_
				4...ec$\underline{\mathtt{a}}$	{3,4}...cab	4...ccb_
91	4...b$\underline{\mathtt{d}}$	{2,3,4,5}...ec	1...bc_	4...bb$\underline{\mathtt{d}}$	{3,4}...cec	1...abc_
				4...db$\underline{\mathtt{d}}$	{2,3,4,5}...cec	1...abc_
92	4...b$\underline{\mathtt{d}}$	{2,3,4,5}...ec	1...bd_	4...bb$\underline{\mathtt{d}}$	{2,3,4,5}...cec	1...dbd_
				4...cb$\underline{\mathtt{d}}$	{2,3,4,5}...aec	1...cbd_
				4...db$\underline{\mathtt{d}}$	{2,3,4,5}...cec	1...cbd_
				4...db$\underline{\mathtt{d}}$	{2,3,4,5}...cec	1...dbd_
93	4...b$\underline{\mathtt{d}}$	{3,4}...ec	3...aa_	4...cb$\underline{\mathtt{d}}$	{3,4}...aec	3...caa_
				4...db$\underline{\mathtt{d}}$	3...cec	3...caa_
94	4...b$\underline{\mathtt{d}}$	3...ec	4...cc_			
125	5...a$\underline{\mathtt{d}}$	{2,3,4,5}...ba	1...bc_	5...ba$\underline{\mathtt{d}}$	{3,4}...cba	1...abc_
				5...ca$\underline{\mathtt{d}}$	{2,3,4,5}...aba	1...dbc_
				5...da$\underline{\mathtt{d}}$	{2,3,4,5}...cba	1...abc_
				5...da$\underline{\mathtt{d}}$	{2,3,4,5}...cba	1...dbc_
126	5...a$\underline{\mathtt{d}}$	{3,4}...ba	3...bc_	5...ca$\underline{\mathtt{d}}$	{3,4}...aba	3...abc_
				5...da$\underline{\mathtt{d}}$	3...cba	3...abc_
127	5...a$\underline{\mathtt{d}}$	{2,3,4,5}...ba	4...cc_	5...ba$\underline{\mathtt{d}}$	{2,3,4,5}...cba	4...bcc_
				5...ca$\underline{\mathtt{d}}$	{2,3,4,5}...aba	4...bcc_
				5...da$\underline{\mathtt{d}}$	{2,3,4,5}...cba	4...bcc_

******************** The following is a procedure for obtaining from a member of $\mathtt{IRR}(\mathtt{n},\mathtt{k})$, the corresponding sets of members of $\mathtt{IRR}(\mathtt{n}+1,\mathtt{k}',\mathtt{k}+1)$ and $\mathtt{IRR}(\mathtt{n},\mathtt{k},\mathtt{k}+1)$ for each value of α. Either or both of these could be empty. Start from a typical member of $\mathtt{IRR}(\mathtt{n},\mathtt{k})$ that can be written as

$$\mathtt{st}\ldots\mathtt{sy}_1\ldots\mathtt{sy}_{\mathtt{k}-1}\underline{\mathtt{sy}_\mathtt{k}} \to \mathtt{st}'\ldots\mathtt{sy}_1'\ldots\mathtt{sy}_\mathtt{k}' \to \mathtt{RHS}. \tag{682}$$

where the computation of the RHS is straightforward and will be assumed to be done in all the following. Therefore $\mathtt{st}\ldots\mathtt{sy}_1\ldots\mathtt{sy}_{\mathtt{k}-1}\underline{\mathtt{sy}_\mathtt{k}}$ determines $\mathtt{st}'\ldots\mathtt{sy}_1'\ldots\mathtt{sy}_\mathtt{k}'$ uniquely and these sequences must appear together in all members of $\mathtt{IRR}(\mathtt{n},\mathtt{k})$. This fact can be used to reduce duplication of information in tables analogous to Tables 2 and 22.

In the derivation of the $\mathtt{IRR}(\mathtt{n}+1,\mathtt{k}')$, for some value of $\mathtt{k}' \leq \mathtt{n}$, from this according to the method involved in Theorem 2.2 (function $\mathtt{F}$), if a branch of the backward search algorithm with a value of α ends in condition 1 this can be represented as

$$\mathtt{st}\ldots\mathtt{sy}_1\ldots\mathtt{sy}_{\mathtt{k}-1}\underline{\mathtt{sy}_\mathtt{k}}\alpha \leftarrow \mathtt{st}^*\ldots\mathtt{sy}_1^*\ldots\underline{\mathtt{sy}_{\mathtt{k}+1}^*}. \tag{683}$$

This is a necessary and sufficient condition for a derived IRR triplet to be found. Suppose the pointer does not reach $\mathtt{sy}_1$ so the minimum length of tape needed for the backward search is $\mathtt{k}'$ where $\mathtt{k}' \leq \mathtt{k}$ because otherwise this allows the possibility that the backward search algorithm ends in condition 2. Therefore $\mathtt{sy_1}$ is unaffected and $\mathtt{sy_1} = \mathtt{sy}_1^*$. (so $\mathtt{sy_1}$ could have several different values without affecting anything else i.e. these results can be put into groups such that within each group the results differ only in the value of $\mathtt{sy_1}$. For each member of the group the range of α and the origins abbreviated to length $\mathtt{k}$ are the same ($\mathtt{sy}_1^*$ is removed)). Therefore the typical member of $\mathtt{IRR}(\mathtt{n}+\mathtt{1}, \mathtt{k}', \mathtt{k}+\mathtt{1})$ derived from this is

$$\mathtt{st}^* \ldots \mathtt{sy}_1^* \ldots \mathtt{sy}_{\mathtt{k}-\mathtt{k}'+2}^* \ldots \underline{\mathtt{sy}_{\mathtt{k}+1}^*} \to \mathtt{st}' \ldots \mathtt{sy}_1' \ldots \mathtt{sy}_{\mathtt{k}-\mathtt{k}'+2}' \ldots \mathtt{sy}_\mathtt{k}'\alpha \to \mathtt{RHS}. \tag{684}$$

If during the derivation of 683 the pointer does not reach the symbol $\mathtt{sy}_2^*$ the result would presumably have been found already from the results for shorter strings (check). If a branch of the backward search algorithm for the above case with a particular value of α does reach $\mathtt{sy_1}$ we get say

$$\mathtt{st} \ldots \mathtt{sy_1} \ldots \mathtt{sy_{k-1}}\underline{\mathtt{sy_k}}\alpha \leftarrow \mathtt{st}'' \ldots \underline{\mathtt{sy_1}} \ldots \mathtt{sy}_\mathtt{k}''\alpha \tag{685}$$

because now α is not reached by the pointer and so is unchanged. In this calculation, when the pointer reaches $\mathtt{sy_1}$ the backward search algorithm stops because another symbol is needed to obtain all possible reverse TM steps from there so this symbol is not changed, as indicated. No member of $\mathtt{IRR}(\mathtt{n}+\mathtt{1}, \mathtt{k}')$ results (because no origin was found and so the LHS was not verified as being reachable) and instead it is necessary to go back by finding what triplet $\mathtt{T} \in \mathtt{IRR}(\mathtt{n}-\mathtt{1}, \mathtt{k_3}, \mathtt{k})$ (that may not be unique) gave rise to 682 via $\mathtt{F}$, and deriving the extended 682 (this is in $\mathtt{IRR}(\mathtt{n}, \mathtt{k}, \mathtt{k}+\mathtt{1})$ and truncates to 682) from $\mathtt{T}$ without truncating the symbol on the left. Then apply $\mathtt{F}$ to the extended 682.

The triplet $\mathtt{T}$ is obtained by first obtaining $\mathtt{T}' \in \mathtt{IRR}(\mathtt{n}-\mathtt{1}, \mathtt{k_1}, \mathtt{k_2})$, that may not be unique, such that $\mathtt{F}$ applied to it yields 682. The origin of $\mathtt{T}'$ is obtained by forward computation from the origin of 682 and finishing when the pointer reaches the symbol at $\mathtt{sy_{k-1}}$. The first TM step must be to the left reaching that point, printing symbol α where $\mathtt{sy_k}$ is. This computation must not have the pointer reach its starting location again, nor the point where $\mathtt{sy_1}$ is, because this is the backward search algorithm running in reverse that would stop in either of these situations. This computation could happen in a lot different ways because the endpoint could be reached by multiple left and right moving sweeps of the pointer. After this, the symbol at the right, that is α is deleted. If there is cycling, the computation must back-track to where the endpoint was last reached. The LHS of $\mathtt{T}'$ must be the LHS of 682 with the last symbol on the right removed, that is α. The RHS of $\mathtt{T}'$ must be obtained by running the TM backwards from the RHS of 682 such that the pointer ends up one place to the left of where it started. This can give non-unique results. Then $\mathtt{T}$ must match $\mathtt{T}'$, where $\mathtt{T}$ has already been found, so $\mathtt{T}$ may not be unique. Then for each of these $\mathtt{T}$'s search for an α such that when $\mathtt{F}$ is applied, it generates a result that matches 682 in $\mathtt{IRR}(\mathtt{n},\mathtt{k},\mathtt{k}+1)$ i.e. without truncation. Then the computation can proceed from there, which may or may not give a result in $\mathtt{IRR}(\mathtt{n}+1,\mathtt{k}+1)$ etc..

I think the general procedure for obtaining the means of getting the IRR($\mathtt{n}$) recursively is something like the following:

After the IRR(3) have been obtained: Obtain IRR(3,2) Table 1 done. From IRR(3,2) obtain IRR(4,2) (Tables 2 and 22) From those entries in IRR(4,2) that could not be completely done because of the restriction on the tape length for calculating all the origins, do this again starting with those IRR(3,2) but keeping the extra symbol giving IRR(4,3). This should account for all the IRR(4) because in running the backward search algorithm to get all the origins from the IRR(3) (starting from $\mathtt{xx}\underline{\mathtt{x}}\alpha$), the furthest left you have to go from the α is 2 spaces because the if the pointer goes one step further it reaches the end and the backward search algorithm stops ($\mathtt{k}=3$). With only one symbol deleted, all the values of α can be obtained from the backward search algorithm because if the pointer gets to the end the search stops. From these IRR(4,3) obtain IRR(5,3) in analogy with how Tables 2 and 22 were obtained. Again, if some of these calculations cannot be completed properly obtain the IRR(5,4) by retaining the extra symbol. ... etc.

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Table 22: Outline IRRs of length $\mathtt{n}$ and their derived outline IRR of length $\mathtt{n}+1$, with the unknown symbols on the left.

set number	IRR($\mathtt{n}$) Origin RHS	α	IRR($\mathtt{n}+1$) Origin RHS
1	$1\ldots\mathtt{b}\underline{\mathtt{a}}$ $1\ldots\mathtt{ba\_}$	$\emptyset$	$\emptyset$
2	$1\ldots\mathtt{b}\underline{\mathtt{a}}$ $1\ldots\mathtt{bd\_}$	$\emptyset$	$\emptyset$
3	$1\ldots\mathtt{b}\underline{\mathtt{a}}$ $2\ldots\mathtt{db\_}$	$\emptyset$	$\emptyset$
4	$1\ldots\mathtt{e}\underline{\mathtt{a}}$ $1\ldots\mathtt{ba\_}$	$\emptyset$	$\emptyset$
5	$1\ldots\mathtt{e}\underline{\mathtt{a}}$ $2\ldots\mathtt{db\_}$	$\emptyset$	$\emptyset$
6	$1\ldots\mathtt{a}\underline{\mathtt{b}}$ $2\ldots\mathtt{ec}$	$\emptyset$	$\emptyset$
7	$1\ldots\mathtt{a}\underline{\mathtt{b}}$ $1\ldots\mathtt{ba\_}$	$\emptyset$	$\emptyset$
8	$1\ldots\mathtt{a}\underline{\mathtt{b}}$ $2\ldots\mathtt{db\_}$	$\emptyset$	$\emptyset$
9	$1\ldots\mathtt{d}\underline{\mathtt{b}}$ $1\ldots\mathtt{ba\_}$	α	$2\ldots\underline{\mathtt{c}}\mathtt{b}\alpha$ RHS
10	$1\ldots\mathtt{d}\underline{\mathtt{b}}$ $1\ldots\mathtt{bd\_}$	α	$2\ldots\underline{\mathtt{c}}\mathtt{b}\alpha$ RHS
11	$1\ldots\mathtt{d}\underline{\mathtt{b}}$ $2\ldots\mathtt{cb\_}$	α	$2\ldots\underline{\mathtt{c}}\mathtt{b}\alpha$ RHS
12	$1\ldots\mathtt{d}\underline{\mathtt{b}}$ $2\ldots\mathtt{db\_}$	α	$2\ldots\underline{\mathtt{c}}\mathtt{b}\alpha$ RHS
13	$1\ldots\mathtt{b}\underline{\mathtt{c}}$ $1\ldots\mathtt{bc\_}$	$\emptyset$	$\emptyset$
14	$1\ldots\mathtt{b}\underline{\mathtt{c}}$ $2\ldots\mathtt{ab\_}$	$\emptyset$	$\emptyset$
15	$1\ldots\mathtt{b}\underline{\mathtt{c}}$ $4\ldots\mathtt{ac\_}$	$\emptyset$	$\emptyset$
16	$1\ldots\mathtt{b}\underline{\mathtt{c}}$ $4\ldots\mathtt{cc\_}$	$\emptyset$	$\emptyset$
17	$1\ldots\mathtt{c}\underline{\mathtt{c}}$ $1\ldots\mathtt{bc\_}$	α	$2\ldots\underline{\mathtt{a}}\mathtt{c}\alpha$ RHS
18	$1\ldots\mathtt{c}\underline{\mathtt{c}}$ $2\ldots\mathtt{db\_}$	α	$2\ldots\underline{\mathtt{a}}\mathtt{c}\alpha$ RHS
19	$1\ldots\mathtt{c}\underline{\mathtt{c}}$ $3\ldots\mathtt{bc\_}$	α	$2\ldots\underline{\mathtt{a}}\mathtt{c}\alpha$ RHS
20	$1\ldots\mathtt{d}\underline{\mathtt{c}}$ $1\ldots\mathtt{bc\_}$	α	$2\ldots\underline{\mathtt{c}}\mathtt{c}\alpha$ RHS
21	$1\ldots\mathtt{d}\underline{\mathtt{c}}$ $2\ldots\mathtt{ab\_}$	α	$2\ldots\underline{\mathtt{c}}\mathtt{c}\alpha$ RHS
22	$1\ldots\mathtt{d}\underline{\mathtt{c}}$ $2\ldots\mathtt{db\_}$	α	$2\ldots\underline{\mathtt{c}}\mathtt{c}\alpha$ RHS
23	$1\ldots\mathtt{d}\underline{\mathtt{c}}$ $3\ldots\mathtt{bc\_}$	α	$2\ldots\underline{\mathtt{c}}\mathtt{c}\alpha$ RHS
24	$1\ldots\mathtt{d}\underline{\mathtt{c}}$ $4\ldots\mathtt{ac\_}$	α	$2\ldots\underline{\mathtt{c}}\mathtt{c}\alpha$ RHS
25	$1\ldots\mathtt{d}\underline{\mathtt{c}}$ $4\ldots\mathtt{cc\_}$	α	$2\ldots\underline{\mathtt{c}}\mathtt{c}\alpha$ RHS

26	2...a$\underline{b}$ 1...bd_	a d e	3...b$\underline{c}$ 2...ab_ 1...b$\underline{a}$ 2...db_ 5...b$\underline{a}$ 2...db_
27	2...a$\underline{b}$ 2...ab_	a d e	3...b$\underline{c}$ 1...bc_ 1...b$\underline{a}$ 1...ba_ 5...b$\underline{a}$ 3...cc
28	2...a$\underline{b}$ 3...bc_	a d e	3...b$\underline{c}$ 4...cc_ 1...b$\underline{a}$ 2...db_ 5...b$\underline{a}$ 3...cb_
29	2...d$\underline{b}$ 3...ca	$\emptyset$	$\emptyset$
30	2...d$\underline{b}$ 4...aa	$\emptyset$	$\emptyset$
31	2...d$\underline{b}$ 1...bd_	a d e	3...b$\underline{c}$ 2...ab_ 1...b$\underline{a}$ 2...db_ 5...b$\underline{a}$ 2...db_
32	2...d$\underline{b}$ 2...ab_	a d e	3...b$\underline{c}$ 1...bc_ 1...b$\underline{a}$ 1...ba_ 5...b$\underline{a}$ 3...cc
33	2...d$\underline{b}$ 2...db_	a d e	3...b$\underline{c}$ 1...bc_ 1...b$\underline{a}$ 1...ba_ 5...b$\underline{a}$ 5...cc
34	2...d$\underline{b}$ 3...cc_	a d e	3...b$\underline{c}$ 4...cc_ 1...b$\underline{a}$ 2...db_ 5...b$\underline{a}$ 3...cb_
35	2...b$\underline{e}$ 3...ca	$\emptyset$	$\emptyset$
36	2...b$\underline{e}$ 1...bd_	a d e α α	3...e$\underline{c}$ 2...ab_ 1...e$\underline{a}$ 2...db_ 5...e$\underline{a}$ 2...db_ 1...$\underline{d}$eα RHS 1...$\underline{e}$eα RHS
37	2...b$\underline{e}$ 2...ab_	a d e α α	3...e$\underline{c}$ 1...bc_ 1...e$\underline{a}$ 1...ba_ 5...e$\underline{a}$ 3...cc 1...$\underline{d}$eα RHS 1...$\underline{e}$eα RHS
38	2...b$\underline{e}$ 2...db_	a d e α α	3...e$\underline{c}$ 1...bc_ 1...e$\underline{a}$ 1...ba_ 5...e$\underline{a}$ 5...cc 1...$\underline{d}$eα RHS 1...$\underline{e}$eα RHS
39	2...c$\underline{e}$ 4...aa	$\emptyset$	$\emptyset$
40	2...c$\underline{e}$ 1...bd_	a d e	3...e$\underline{c}$ 2...ab_ 1...e$\underline{a}$ 2...db_ 5...e$\underline{a}$ 2...db_
41	2...c$\underline{e}$ 3...cc_	a d e	3...e$\underline{c}$ 4...cc_ 1...e$\underline{a}$ 2...db_ 5...e$\underline{a}$ 3...cb_
42	2...d$\underline{e}$ 3...ca	$\emptyset$	$\emptyset$
43	2...d$\underline{e}$ 4...aa	$\emptyset$	$\emptyset$
44	2...d$\underline{e}$ 1...bd_	a d e	3...e$\underline{c}$ 2...ab_ 1...e$\underline{a}$ 2...db_ 5...e$\underline{a}$ 2...db_
45	2...d$\underline{e}$ 2...ab_	a d e	3...e$\underline{c}$ 1...bc_ 1...e$\underline{a}$ 1...ba_ 5...e$\underline{a}$ 3...cc
46	2...d$\underline{e}$ 2...db_	a d e	3...e$\underline{c}$ 1...bc_ 1...e$\underline{a}$ 1...ba_ 5...e$\underline{a}$ 5...cc
47	2...d$\underline{e}$ 3...cc_	a d	3...e$\underline{c}$ 4...cc_ 1...e$\underline{a}$ 2...db_

		e	5...e$\underline{\text{a}}$	3...cb␣
48	3...a$\underline{\text{b}}$ 1...bd␣	a	1...b$\underline{\text{c}}$	2...ab␣
		b	4...b$\underline{\text{a}}$	3...cd
		c	2...b$\underline{\text{e}}$	3...ca
		e	5...b$\underline{\text{b}}$	2...db␣
		α	5...$\underline{\text{c}}$bα	RHS
		α	5...$\underline{\text{e}}$bα	RHS
49	3...a$\underline{\text{b}}$ 2...ab␣	a	1...b$\underline{\text{c}}$	1...bc␣
		b	4...b$\underline{\text{a}}$	3...bc␣
		c	2...b$\underline{\text{e}}$	1...bd␣
		e	5...b$\underline{\text{b}}$	3...cc
		α	5...$\underline{\text{c}}$bα	RHS
		α	5...$\underline{\text{e}}$bα	RHS
50	3...a$\underline{\text{b}}$ 3...bc␣	a	1...b$\underline{\text{c}}$	4...cc␣
		b	4...b$\underline{\text{a}}$	2...ab␣
		c	2...b$\underline{\text{e}}$	3...cb␣
		e	5...b$\underline{\text{b}}$	RHS
		α	5...$\underline{\text{c}}$bα	RHS
		α	5...$\underline{\text{e}}$bα	RHS
51	3...d$\underline{\text{b}}$ 3...ca	$\emptyset$	$\emptyset$	
52	3...d$\underline{\text{b}}$ 4...aa	$\emptyset$	$\emptyset$	
53	3...d$\underline{\text{b}}$ 1...bd␣	a	1...b$\underline{\text{c}}$	2...ab␣
		b	4...b$\underline{\text{a}}$	3...cd
		c	2...b$\underline{\text{e}}$	3...ca
		e	5...b$\underline{\text{b}}$	2...db␣
54	3...d$\underline{\text{b}}$ 2...ab␣	a	1...b$\underline{\text{c}}$	1...bc␣
		b	4...b$\underline{\text{a}}$	3...bc␣
		c	2...b$\underline{\text{e}}$	1...bd␣
		e	5...b$\underline{\text{b}}$	3...cc
55	3...d$\underline{\text{b}}$ 2...db␣	a	1...b$\underline{\text{c}}$	1...bc␣
		b	4...b$\underline{\text{a}}$	3...bc␣
		c	2...b$\underline{\text{e}}$	1...bd␣
		e	5...b$\underline{\text{b}}$	5...cc
56	3...d$\underline{\text{b}}$ 3...cc␣	a	1...b$\underline{\text{c}}$	4...cc␣
		b	4...b$\underline{\text{a}}$	2...ab␣
		c	2...b$\underline{\text{e}}$	2...ab␣
		e	5...b$\underline{\text{b}}$	3...cb␣
57	3...b$\underline{\text{c}}$ 1...bc␣	a	1...c$\underline{\text{c}}$	2...db␣
		b	4...c$\underline{\text{a}}$	2...db␣
		c	2...c$\underline{\text{e}}$	4...aa
		e	5...c$\underline{\text{b}}$	2...cb␣
		α	3...$\underline{\text{e}}$cα	RHS
58	3...b$\underline{\text{c}}$ 2...ab␣	a	1...c$\underline{\text{c}}$	1...bc␣
		b	4...c$\underline{\text{a}}$	3...bc␣
		c	2...c$\underline{\text{e}}$	1...bd␣
		e	5...c$\underline{\text{b}}$	3...cc
		α	3...$\underline{\text{e}}$cα	RHS
59	3...b$\underline{\text{c}}$ 4...ac␣	a	1...c$\underline{\text{c}}$	3...bc␣
		b	4...c$\underline{\text{a}}$	4...cb␣
		c	2...c$\underline{\text{e}}$	3...cc␣
		e	5...c$\underline{\text{b}}$	5...ca␣
		α	3...$\underline{\text{e}}$cα	RHS
60	3...b$\underline{\text{c}}$ 4...cc␣	a	1...c$\underline{\text{c}}$	3...bc␣
		b	4...c$\underline{\text{a}}$	4...cb␣
		c	2...c$\underline{\text{e}}$	3...cc␣
		e	5...c$\underline{\text{b}}$	5...ca␣
		α	3...$\underline{\text{e}}$cα	RHS
61	3...e$\underline{\text{c}}$ 1...bc␣	a	1...c$\underline{\text{c}}$	2...db␣
		b	4...c$\underline{\text{a}}$	2...db␣
		c	2...c$\underline{\text{e}}$	4...aa

		e	5...c$\underline{b}$	2...cb_
62	3...e$\underline{c}$ 2...ab_	a	1...c$\underline{c}$	1...bc_
		b	4...c$\underline{a}$	3...bc_
		c	2...c$\underline{e}$	1...bd_
		e	5...c$\underline{b}$	3...cc
63	3...e$\underline{c}$ 4...cc_	a	1...c$\underline{c}$	3...bc_
		b	4...c$\underline{a}$	4...cb_
		c	2...c$\underline{e}$	3...cc_
		e	5...c$\underline{b}$	5...ca_
64	3...a$\underline{d}$ 3...ca	$\emptyset$	$\emptyset$	
65	3...a$\underline{d}$ 1...bc_	a	1...d$\underline{c}$	2...db_
		b	4...d$\underline{a}$	2...db_
		c	2...d$\underline{e}$	4...aa
		e	5...d$\underline{b}$	2...cb_
		α	5...$\underline{c}$dα	RHS
		α	5...$\underline{e}$dα	RHS
66	3...a$\underline{d}$ 1...bd_	a	1...d$\underline{c}$	2...ab_
		b	4...d$\underline{a}$	3...cd
		c	2...d$\underline{e}$	3...ca
		e	5...d$\underline{b}$	2...db_
		α	5...$\underline{c}$dα	RHS
		α	5...$\underline{e}$dα	RHS
67	3...b$\underline{d}$ 3...ca	$\emptyset$	$\emptyset$	
68	3...b$\underline{d}$ 1...bc_	a	1...d$\underline{c}$	2...db_
		b	4...d$\underline{a}$	2...db_
		c	2...d$\underline{e}$	4...aa
		e	5...d$\underline{b}$	2...cb_
		α	3...$\underline{e}$dα	RHS
69	3...b$\underline{d}$ 1...bd_	a	1...d$\underline{c}$	2...ab_
		b	4...d$\underline{a}$	3...cd
		c	2...d$\underline{e}$	3...ca
		e	5...d$\underline{b}$	2...db_
		α	3...$\underline{e}$dα	RHS
70	3...b$\underline{d}$ 3...aa_	a	1...d$\underline{c}$	4...ac_
		b	4...d$\underline{a}$	3...bc_
		c	2...d$\underline{e}$	2...db_
		e	5...d$\underline{b}$	3...ab_
		α	3...$\underline{e}$dα	RHS
71	3...b$\underline{d}$ 4...cc_	a	1...d$\underline{c}$	3...bc_
		b	4...d$\underline{a}$	4...cb_
		c	2...d$\underline{e}$	3...cc_
		e	5...d$\underline{b}$	5...ca_
		α	3...$\underline{e}$dα	RHS
72	3...d$\underline{d}$ 4...aa	$\emptyset$	$\emptyset$	
73	3...d$\underline{d}$ 1...bd_	a	1...d$\underline{c}$	2...ab_
		b	4...d$\underline{a}$	3...cd
		c	2...d$\underline{e}$	3...ca
		e	5...d$\underline{b}$	2...db_
74	3...d$\underline{d}$ 2...ab_	a	1...d$\underline{c}$	1...bc_
		b	4...d$\underline{a}$	3...bc_
		c	2...d$\underline{e}$	1...bd_
		e	5...d$\underline{b}$	3...cc
75	3...d$\underline{d}$ 3...cc_	a	1...d$\underline{c}$	4...cc_
		b	4...d$\underline{a}$	2...ab_
		c	2...d$\underline{e}$	2...ab_
		e	5...d$\underline{b}$	3...cb_
76	4...b$\underline{a}$ 3...cd	$\emptyset$	$\emptyset$	
77	4...b$\underline{a}$ 2...ab_	a	5...a$\underline{d}$	1...bc_
		c	2...a$\underline{b}$	1...bd_
		c	3...a$\underline{b}$	1...bd_

		d	1...a$\underline{b}$	1...ba␣
		α	4...$\underline{b}$aα	RHS
78	4...b$\underline{a}$ 3...bc␣	a	5...a$\underline{d}$	4...cc␣
		c	2...a$\underline{b}$	2...ab␣
		c	3...a$\underline{b}$	2...ab␣
		d	1...a$\underline{b}$	2...db␣
		α	4...$\underline{b}$aα	RHS
79	4...c$\underline{a}$ 2...db␣	a	5...a$\underline{d}$	1...bc␣
		c	2...a$\underline{b}$	1...bd␣
		c	3...a$\underline{b}$	1...bd␣
		d	1...a$\underline{b}$	1...ba␣
		α	3...$\underline{a}$aα	RHS
80	4...c$\underline{a}$ 3...bc␣	a	5...a$\underline{d}$	4...cc␣
		c	2...a$\underline{b}$	2...ab␣
		c	3...a$\underline{b}$	2...ab␣
		d	1...a$\underline{b}$	2...db␣
		α	3...$\underline{a}$aα	RHS
81	4...c$\underline{a}$ 4...cb␣	a	5...a$\underline{d}$	3...bc␣
		c	2...a$\underline{b}$	3...bc␣
		c	3...a$\underline{b}$	3...bc␣
		d	1...a$\underline{b}$	2...ec
		α	3...$\underline{a}$aα	RHS
82	4...d$\underline{a}$ 3...cd	$\emptyset$	$\emptyset$	
83	4...d$\underline{a}$ 2...ab␣	a	5...a$\underline{d}$	1...bc␣
		c	2...a$\underline{b}$	1...bd␣
		c	3...a$\underline{b}$	1...bd␣
		d	1...a$\underline{b}$	1...ba␣
84	4...d$\underline{a}$ 2...db␣	a	5...a$\underline{d}$	1...bc␣
		c	2...a$\underline{b}$	1...bd␣
		c	3...a$\underline{b}$	1...bd␣
		d	1...a$\underline{b}$	1...ba␣
85	4...d$\underline{a}$ 3...bc␣	a	5...a$\underline{d}$	4...cc␣
		c	2...a$\underline{b}$	2...ab␣
		c	3...a$\underline{b}$	2...ab␣
		d	1...a$\underline{b}$	2...db␣
86	4...d$\underline{a}$ 4...cb␣	a	5...a$\underline{d}$	3...bc␣
		c	2...a$\underline{b}$	3...bc␣
		c	3...a$\underline{b}$	3...bc␣
		d	1...a$\underline{b}$	2...ec
87	4...a$\underline{d}$ 3...ca	$\emptyset$	$\emptyset$	
88	4...a$\underline{d}$ 1...bc␣	a	5...d$\underline{d}$	2...db␣
		c	2...d$\underline{b}$	4...aa
		c	3...d$\underline{b}$	4...aa
		d	1...d$\underline{b}$	2...cb␣
89	4...a$\underline{d}$ 1...bd␣	a	5...d$\underline{d}$	2...ab␣
		c	2...d$\underline{b}$	3...ca
		c	3...d$\underline{b}$	3...ca
		d	1...d$\underline{b}$	2...db␣
90	4...b$\underline{d}$ 3...ca	$\emptyset$	$\emptyset$	
91	4...b$\underline{d}$ 1...bc␣	a	5...d$\underline{d}$	2...db␣
		c	2...d$\underline{b}$	4...aa
		c	3...d$\underline{b}$	4...aa
		d	1...d$\underline{b}$	2...cb␣
		α	4...$\underline{b}$dα	RHS
92	4...b$\underline{d}$ 1...bd␣	a	5...d$\underline{d}$	2...ab␣
		c	2...d$\underline{b}$	3...ca
		c	3...d$\underline{b}$	3...ca
		d	1...d$\underline{b}$	2...db␣
		α	4...$\underline{b}$dα	RHS
93	4...b$\underline{d}$ 3...aa␣	a	5...d$\underline{d}$	4...ac␣

		c	2...d$\underline{b}$ 2...db_
		c	3...d$\underline{b}$ 2...db_
		d	1...d$\underline{b}$ 1...bd_
		α	4...$\underline{b}$dα RHS
94	4...b$\underline{d}$ 4...cc_	a	5...d$\underline{d}$ 3...bc_
		c	2...d$\underline{b}$ 3...cc_
		c	3...d$\underline{b}$ 3...cc_
		d	1...d$\underline{b}$ 2...db_
		α	4...$\underline{b}$dα RHS
95	4...d$\underline{d}$ 4...aa	$\emptyset$	$\emptyset$
96	4...d$\underline{d}$ 1...bd_	a	5...d$\underline{d}$ 2...ab_
		c	2...d$\underline{b}$ 3...ca
		c	3...d$\underline{b}$ 3...ca
		d	1...d$\underline{b}$ 2...db_
97	4...d$\underline{d}$ 2...ab_	a	5...d$\underline{d}$ 1...bc_
		c	2...d$\underline{b}$ 1...bd_
		c	3...d$\underline{b}$ 1...bd_
		d	1...d$\underline{b}$ 1...ba_
98	4...d$\underline{d}$ 3...cc_	a	5...d$\underline{d}$ 4...cc_
		c	2...d$\underline{b}$ 2...ab_
		c	3...d$\underline{b}$ 2...ab_
		d	1...d$\underline{b}$ 2...db_
99	5...b$\underline{a}$ 3...cc	$\emptyset$	$\emptyset$
100	5...b$\underline{a}$ 5...cc	$\emptyset$	$\emptyset$
101	5...b$\underline{a}$ 2...db_	c	3...a$\underline{d}$ 1...bd_
		c	4...a$\underline{d}$ 1...bd_
102	5...b$\underline{a}$ 3...ab_	c	3...a$\underline{d}$ 3...ca
		c	4...a$\underline{d}$ 3...ca
103	5...b$\underline{a}$ 3...cb_	c	3...a$\underline{d}$ 1...bc_
		c	4...a$\underline{d}$ 1...bc_
104	5...e$\underline{a}$ 3...cc	$\emptyset$	$\emptyset$
105	5...e$\underline{a}$ 5...cc	$\emptyset$	$\emptyset$
106	5...e$\underline{a}$ 2...db_	c	3...a$\underline{d}$ 1...bd_
		c	4...a$\underline{d}$ 1...bd_
107	5...e$\underline{a}$ 3...cb_	c	3...a$\underline{d}$ 1...bc_
		c	4...a$\underline{d}$ 1...bc_
108	5...b$\underline{b}$ 3...cc	$\emptyset$	$\emptyset$
109	5...b$\underline{b}$ 5...cc	$\emptyset$	$\emptyset$
110	5...b$\underline{b}$ 2...db_	c	3...b$\underline{d}$ 1...bd_
		c	4...b$\underline{d}$ 1...bd_
111	5...b$\underline{b}$ 3...ab_	c	3...b$\underline{d}$ 3...ca
		c	4...b$\underline{d}$ 3...ca
112	5...b$\underline{b}$ 3...cb_	c	3...b$\underline{d}$ 1...bc_
		c	4...b$\underline{d}$ 1...bc_
113	5...c$\underline{b}$ 3...cc	$\emptyset$	$\emptyset$
114	5...c$\underline{b}$ 5...cc	$\emptyset$	$\emptyset$
115	5...c$\underline{b}$ 1...bd_	c	3...b$\underline{d}$ 3...ca
		c	4...b$\underline{d}$ 3...ca
116	5...c$\underline{b}$ 2...cb_	c	3...b$\underline{d}$ 1...bd_
		c	4...b$\underline{d}$ 1...bd_
117	5...c$\underline{b}$ 5...ca_	c	3...b$\underline{d}$ 3...aa_
		c	4...b$\underline{d}$ 3...aa_
118	5...d$\underline{b}$ 3...cc	$\emptyset$	$\emptyset$
119	5...d$\underline{b}$ 5...cc	$\emptyset$	$\emptyset$
120	5...d$\underline{b}$ 2...cb_	c	3...b$\underline{d}$ 1...bd_
		c	4...b$\underline{d}$ 1...bd_
121	5...d$\underline{b}$ 2...db_	c	3...b$\underline{d}$ 1...bd_
		c	4...b$\underline{d}$ 1...bd_
122	5...d$\underline{b}$ 3...ab_	c	3...b$\underline{d}$ 3...ca

		c	4...b$\underline{d}$	3...ca
123	5...d$\underline{b}$ 3...cb_	c	3...b$\underline{d}$	1...bc_
		c	4...b$\underline{d}$	1...bc_
124	5...d$\underline{b}$ 5...ca_	c	3...b$\underline{d}$	3...aa_
		c	4...b$\underline{d}$	3...aa_
125	5...a$\underline{d}$ 1...bc_	c	3...d$\underline{d}$	4...aa
		c	4...d$\underline{d}$	4...aa
		α	4...$\underline{e}$dα	RHS
126	5...a$\underline{d}$ 3...bc_	c	3...d$\underline{d}$	2...ab_
		c	4...d$\underline{d}$	2...ab_
		α	4...$\underline{e}$dα	RHS
127	5...a$\underline{d}$ 4...cc_	c	3...d$\underline{d}$	3...cc_
		c	4...d$\underline{d}$	3...cc_
		α	4...$\underline{e}$dα	RHS
128	5...d$\underline{d}$ 1...bc_	c	3...d$\underline{d}$	4...aa
		c	4...d$\underline{d}$	4...aa
129	5...d$\underline{d}$ 2...ab_	c	3...d$\underline{d}$	1...bd_
		c	4...d$\underline{d}$	1...bd_
130	5...d$\underline{d}$ 2...db_	c	3...d$\underline{d}$	1...bd_
		c	4...d$\underline{d}$	1...bd_
131	5...d$\underline{d}$ 3...bc_	c	3...d$\underline{d}$	2...ab_
		c	4...d$\underline{d}$	2...ab_
132	5...d$\underline{d}$ 4...ac_	c	3...d$\underline{d}$	3...cc_
		c	4...d$\underline{d}$	3...cc_
133	5...d$\underline{d}$ 4...cc_	c	3...d$\underline{d}$	3...cc_
		c	4...d$\underline{d}$	3...cc_

In the same way that Table 8 was constructed from Table 2, the following table was constructed from Table 22 using an iterative process. The set of LHS states was obtained by a similar iterative argument by ensuring that any set in column 2 with LHS state x implies that state x is also associated with the same set when it appears in column 1. The result was obtained after this iteration has converged i.e. no change was obtained after one cycle.

Table 23: Relations between sets of IRR outlines of types LL and LR under F

Original set of IRR outlines	Set of sets of IRR outlines derived by F	Set of LHS states
$\{1\}$	$\emptyset$	$\{2,3,4,5\}$
$\{1,4\}$	$\emptyset$	$\{2,3,4,5\}$
$\{2\}$	$\emptyset$	$\{4\}$
$\{3\}$	$\emptyset$	$\{2,3,4,5\}$
$\{3,5\}$	$\emptyset$	$\{3,4,5\}$
$\{4\}$	$\emptyset$	$\{3,4\}$
$\{5\}$	$\emptyset$	$\{2,3,4,5\}$
$\{6\}$	$\emptyset$	$\{3,4\}$
$\{7\}$	$\emptyset$	$\{2,3,4,5\}$
$\{8\}$	$\emptyset$	$\{2,3,4,5\}$
$\{9\}$	$\emptyset$	$\{3,4,5\}$
$\{10\}$	$\emptyset$	$\{3,4\}$
$\{11\}$	$\emptyset$	$\{2,3,4,5\}$
$\{12\}$	$\emptyset$	$\{2,3,4,5\}$
$\{13,57\}$	$\{18\},\{79\},\{39\},\{116\}$	$\{2,3,4,5\}$
$\{13,57,61\}$	$\{18\},\{79\},\{39\},\{116\}$	$\{2,3,4,5\}$
$\{14,58,62\}$	$\{17\},\{80\},\{40\},\{113\}$	$\{3,4,5\}$
$\{14,58\}$	$\{17\},\{80\},\{40\},\{113\}$	$\{2,3,4,5\}$
$\{15,59\}$	$\{19\},\{81\},\{41\},\{117\}$	$\{4\}$
$\{16,60\}$	$\{19\},\{81\},\{41\},\{117\}$	$\{3,4\}$
$\{16,60,63\}$	$\{19\},\{81\},\{41\},\{117\}$	$\{3\}$
$\{17\}$	$\emptyset$	$\{2,3,4,5\}$
$\{18\}$	$\emptyset$	$\{2,3,4,5\}$
$\{19\}$	$\emptyset$	$\{3,4\}$
$\{20,128\}$	$\{72,95\}$	$\{3,4,5\}$
$\{21,129\}$	$\{73,96\}$	$\{2,3,4,5\}$
$\{22,130\}$	$\{73,96\}$	$\{2,3,4,5\}$
$\{23,131\}$	$\{74,97\}$	$\{3\}$
$\{24,132\}$	$\{75,98\}$	$\{3,4\}$
$\{25,133\}$	$\{75,98\}$	$\{2,3,4,5\}$
$\{26,48\}$	$\{14,58\},\{76\},\{35\},\{3\},\{101,110\}$	$\{2,3,4,5\}$
$\{27,49\}$	$\{13,57\},\{78\},\{36\},\{1\},\{99,108\}$	$\{2,3,4,5\}$
$\{28,50\}$	$\{16,60\},\{77\},\{37\},\{3\},\{103,112\}$	$\{3,4\}$
$\{29,42,51\}$	$\emptyset$	$\{2,3,4,5\}$
$\{30,43,52\}$	$\emptyset$	$\{2,3,4,5\}$
$\{31,44,53\}$	$\{14,58,62\},\{76\},\{35\},\{3,5\},\{101,106,110\}$	$\{3,4,5\}$
$\{32,45,54\}$	$\{13,57,61\},\{78\},\{36\},\{1,4\},\{99,104,108\}$	$\{2,3,4,5\}$
$\{33,46,55\}$	$\{13,57,61\},\{78\},\{36\},\{1,4\},\{100,105,109\}$	$\{3,4\}$
$\{34,47,56\}$	$\{16,60,63\},\{77\},\{37\},\{3,5\},\{103,107,112\}$	$\{3\}$
$\{35\}$	$\emptyset$	$\{2,3,4,5\}$

$\{36\}$	$\{62\},\{5\},\{106\}$	$\{2,3,4,5\}$
$\{37\}$	$\{61\},\{4\},\{104\}$	$\{3,4\}$
$\{38\}$	$\{61\},\{4\},\{105\}$	$\{4\}$
$\{39\}$	$\emptyset$	$\{2,3,4,5\}$
$\{40\}$	$\{62\},\{5\},\{106\}$	$\{2,3,4,5\}$
$\{41\}$	$\{63\},\{5\},\{107\}$	$\{3,4\}$
$\{61\}$	$\{18\},\{79\},\{39\},\{116\}$	$\{3,4\}$
$\{62\}$	$\{17\},\{80\},\{40\},\{113\}$	$\{2,3,4,5\}$
$\{63\}$	$\{19\},\{81\},\{41\},\{117\}$	$\{3,4\}$
$\{64,67,87,90\}$	$\emptyset$	$\{4\}$
$\{65,88\}$	$\{22,130\},\{84\},\{30,43,52\},\{11\},\{120\}$	$\{3,4\}$
$\{65,68,88,91\}$	$\{22,130\},\{84\},\{30,43,52\},\{11\},\{120\}$	$\{3,4\}$
$\{66,89\}$	$\{21,129\},\{82\},\{29,42,51\},\{12\},\{121\}$	$\{2,3,4,5\}$
$\{66,69,89,92\}$	$\{21,129\},\{82\},\{29,42,51\},\{12\},\{121\}$	$\{2,3,4,5\}$
$\{67,90\}$	$\emptyset$	$\{2,3,4\}$
$\{68,91\}$	$\{22,130\},\{84\},\{30,43,52\},\{11\},\{120\}$	$\{2,3,4,5\}$
$\{69,92\}$	$\{21,129\},\{82\},\{29,42,51\},\{12\},\{121\}$	$\{2,3,4,5\}$
$\{70,93\}$	$\{24,132\},\{85\},\{33,46,55\},\{10\},\{122\}$	$\{3,4\}$
$\{71,94\}$	$\{23,131\},\{86\},\{34,47,56\},\{12\},\{124\}$	$\{3\}$
$\{72,95\}$	$\emptyset$	$\{2,3,4,5\}$
$\{73,96\}$	$\{21,129\},\{82\},\{29,42,51\},\{12\},\{121\}$	$\{2,3,4,5\}$
$\{74,97\}$	$\{20,128\},\{85\},\{31,44,53\},\{9\},\{118\}$	$\{3,4\}$
$\{75,98\}$	$\{25,133\},\{83\},\{32,45,54\},\{12\},\{123\}$	$\{2,3,4,5\}$
$\{76\}$	$\emptyset$	$\{2,3,4,5\}$
$\{77\}$	$\{125\},\{26,48\},\{7\}$	$\{3,4\}$
$\{78\}$	$\{127\},\{27,49\},\{8\}$	$\{2,3,4,5\}$
$\{79\}$	$\{125\},\{26,48\},\{7\}$	$\{2,3,4,5\}$
$\{80\}$	$\{127\},\{27,49\},\{8\}$	$\{2,3,4,5\}$
$\{81\}$	$\{126\},\{28,50\},\{6\}$	$\{3,4\}$
$\{82\}$	$\emptyset$	$\{2,3,4,5\}$
$\{83\}$	$\{125\},\{26,48\},\{7\}$	$\{2,3,4,5\}$
$\{84\}$	$\{125\},\{26,48\},\{7\}$	$\{2,3,4,5\}$
$\{85\}$	$\{127\},\{27,49\},\{8\}$	$\{3,4,5\}$
$\{86\}$	$\{126\},\{28,50\},\{6\}$	$\{3\}$
$\{99,108\}$	$\emptyset$	$\{2,3,4,5\}$
$\{99,104,108\}$	$\emptyset$	$\{2,3,4,5\}$
$\{100,105,109\}$	$\emptyset$	$\{3,4\}$
$\{101,110\}$	$\{66,69,89,92\}$	$\{2,3,4,5\}$
$\{101,106,110\}$	$\{66,69,89,92\}$	$\{3,4,5\}$
$\{102,111\}$	$\{64,67,87,90\}$	$\{4\}$
$\{103,112\}$	$\{65,68,88,91\}$	$\{3,4\}$
$\{103,107,112\}$	$\{65,68,88,91\}$	$\{3\}$
$\{104\}$	$\emptyset$	$\{3,4\}$
$\{105\}$	$\emptyset$	$\{4\}$

{106}	{66, 89}	{2, 3, 4, 5}
{107}	{65, 88}	{3, 4}
{113}	∅	{2, 3, 4, 5}
{114}	∅	{2}
{115}	{67, 90}	{2}
{116}	{69, 92}	{2, 3, 4, 5}
{117}	{70, 93}	{3, 4}
{118}	∅	{3, 4}
{119}	∅	{5}
{120}	{69, 92}	{2, 3, 4, 5}
{121}	{69, 92}	{2, 3, 4, 5}
{122}	{67, 90}	{3, 4}
{123}	{68, 91}	{2, 3, 4, 5}
{124}	{70, 93}	{3}
{125}	{72, 95}	{2, 3, 4, 5}
{126}	{74, 97}	{3, 4}
{127}	{75, 98}	{2, 3, 4, 5}

From these results, Table 25 of all the numbers (p) of steps needed to get to and from each set (by any path) to each NSCC (`S1`,`S2`, and `S3`) were found. This table in in two parts that were for convenience put side by side, the path lengths to each NSCC and the path lengths from each NSCC. This was done to better understand the context of the NSCC's in the whole directed graph and facilitate the calculation of the frequencies of the IRR. The first step is to put in the Table 25 the zeros representing membership of the NSCC's. Then for the first half i.e. paths to each NSCC, ∅ was put for each set and NSCC where the set has ∅ in column 2 of Table 23. Then for each half of Table 25 separately, an iterative process was carried out in which for each cycle, every set was considered once separately in order to update the body of the appropriate half of the table by adding additional values. Cycles of the process were repeated until there were no changes after a complete cycle. Near the end, this calculation can be speeded up by noticing which entries were changed in the last cycle, and it is only the consequences of these changes that have to be followed up in the next cycle. The content of Table 25 had to be chosen in order to ensure convergence, i.e. the eventual reaching a condition where there are no changes in one complete cycle. This required the avoiding of counting path lengths within any NSCC that can increase without limit.

For the first half it is easy to show that updating can be done as follows: go to the row corresponding to a set in Table 23 and to the corresponding entries (`x`) in the middle column excluding those in the same SCC to find the entries in Table 25 for each NSCC arrived at. Add 1 to them and take the union over all the x's and union it with the already known values. If any of the these are zero (put in at stage 1), any other entry for that NSCC must be deleted leaving only the zero, because only first arrival to each NSCC is considered. As an example, consider the path lengths from $\{\mathtt{28,50}\}$ ($\in$ S3) to the NSCC's. The values are currently: -,{2},{0} for paths to S1,S2 and S3 respectively, where - means no information, and $\emptyset$ means there are no paths from there. The values for the sets derived from the set $\{\mathtt{28,50}\}$, except those in S3, are as follows for paths to the NSCC's:

set	S1	S2	S3
{77}	-	{1}	-
{37}	-	-	-
{3}	$\emptyset$	$\emptyset$	$\emptyset$
{103, 112}	{3}	{2}	-

Therefore the combined result for $\{\mathtt{28,50}\}$ is {4},{2,3},{0}. Note that this is not the final result, because the results on which this one depends were updated subsequently. During this calculation it was noted that there were more and more cases where it could be deduced that there are no possible paths from a set to an NSCC and $\emptyset$ was put in the appropriate cells. This happened when all possible paths led to $\emptyset$ as the set of paths from there to an NSCC member.

I also constructed the second half of Table 25 giving the numbers of steps taken to arrive in the set starting from each NSCC. This was started in the same way, but the iterative cycle now takes each set in Table 23 and deduces new values (by adding 1) to be included in the set of values corresponding to each of the x's unless the original set and any of the x's are in the same NSCC in which case that x is ignored. This exception is to avoid the count going interminably round within the NSCC's. Also if one of the x's is in S1, S2 or S3, this is indicated by a zero, and then no other values are recorded in that cell. For example, $\{\mathtt{69,92}\}$ is in S1 and has values {0},{2,3},{4,6} for S1, S2 and S3 respectively, and the x's are $\{\mathtt{21,129}\}$,$\{\mathtt{82}\}$, $\{\mathtt{29,42,51}\}$,$\{\mathtt{12}\}$, and $\{\mathtt{121}\}$ of which $\{\mathtt{21,129}\}$ and $\{\mathtt{121}\}$ are also in S1, so the other three have the values {1},{3,4},{5,7} included in their sets of values for S1, S2 and S3 respectively, being one more step away from the NSCC's than $\{\mathtt{69,92}\}$ is.

After these results had converged, any cells with no information can have no path from the set to the NSCC member, so they were assigned $\emptyset$.

All the results in Table 25 could be extended to giving the frequencies for each of the lengths found and which sets they arrive at in the NSCC's.

Table 25 records the lengths of all paths from the last node in (a) the SCC containing the set or (b) each NSCC to the first node in (a) each NSCC or (b) SCC containing the set. Here a dash indicates a run of values without any omissions.

Table 25: Lengths of all paths between the SCC's and the NSCC's

set	NSCC	$\to$ S1	$\to$ S2	$\to$ S3	S1 $\to$	S2 $\to$	S3 $\to$
{1}	none	∅	∅	∅	∅	{1}	{5}
{1, 4}	none	∅	∅	∅	∅	{1}	{4}
{2}	none	∅	∅	∅	∅	∅	∅
{3}	none	∅	∅	∅	∅	{1}	{1,3}
{3, 5}	none	∅	∅	∅	∅	∅	{4}
{4}	none	∅	∅	∅	∅	∅	{2}
{5}	none	∅	∅	∅	∅	{1}	{1,5,6}
{6}	none	∅	∅	∅	∅	∅	{1}
{7}	none	∅	∅	∅	∅	{1}	{2,4,6}
{8}	none	∅	∅	∅	∅	{1}	{4,5,6}
{9}	none	∅	∅	∅	∅	∅	{3}
{10}	none	∅	∅	∅	∅	∅	{3}
{11}	none	∅	∅	∅	∅	{1}	{3}
{12}	none	∅	∅	∅	{1}	{1,3,4}	{5,6,7,8}
{13, 57}	S2	{2}	{0}	∅	∅	{0}	∅
{13, 57, 61}	S2	{2}	{0}	∅	∅	{0}	{4}
{14, 58, 62}	none	{4}	{1}	∅	∅	∅	{4}
{14, 58}	S2	∅	{0}	∅	∅	{0}	∅
{15, 59}	none	{5,6,7,9}	{4,5,6}	{1}	∅	∅	∅
{16, 60}	S3	{6,7}	{4,5}	{0}	∅	∅	{0}
{16, 60, 63}	none	{5,6,7,9}	{4,5,6}	{1}	∅	∅	∅
{17}	none	∅	∅	∅	∅	{1}	{5}
{18}	none	∅	∅	∅	∅	{1}	{3,5}
{19}	none	∅	∅	∅	∅	∅	{1}
{20, 128}	none	∅	∅	∅	∅	∅	{3}
{21, 129}	S1	{0}	∅	∅	{0}	{3}	{5,6,7,8}
{22, 130}	none	{1}	∅	∅	∅	{1}	{3}
{23, 131}	none	{5,7}	{3,4}	∅	∅	∅	∅
{24, 132}	none	∅	{1}	∅	∅	∅	{3}
{25, 133}	S2	∅	{0}	∅	∅	{0}	∅
{26, 48}	S2	{3}	{0}	∅	∅	{0}	{2}
{27, 49}	S2	∅	{0}	∅	∅	{0}	{4}
{28, 50}	S3	{4,5}	{2,3}	{0}	∅	∅	{0}
{29, 42, 51}	none	∅	∅	∅	{1}	{3,4}	{5,6,7,8}
{30, 43, 52}	none	∅	∅	∅	∅	{1}	{3}
{31, 44, 53}	none	{3,5}	{2}	∅	∅	∅	{3}
{32, 45, 54}	S2	∅	{0}	∅	∅	{0}	∅
{33, 46, 55}	none	{3,4}	{1}	∅	∅	∅	{3}
{34, 47, 56}	none	{4-8,10}	{2,3,5,6,7}	{2}	∅	∅	∅
{35}	none	∅	∅	∅	∅	{1}	{3,4}
{36}	S2	{3}	{0}	∅	∅	{0}	{4}

{37}	none	{3}	{2}	∅	∅	∅	{1}
{38}	none	{3}	{2}	∅	∅	∅	∅
{39}	none	∅	∅	∅	∅	{1}	{3,5}
{40}	S2	{3}	{0}	∅	∅	{0}	{5}
{41}	S3	{4}	{3}	{0}	∅	∅	{0}
{61}	none	{2}	{1}	∅	∅	∅	{2}
{62}	S2	∅	{0}	∅	∅	{0}	∅
{63}	S3	{6,7}	{4}	{0}	∅	∅	{0}
{64, 67, 87, 91}	none	∅	∅	∅	∅	∅	∅
{65, 88}	none	{2}	{1}	∅	∅	∅	{2}
{65, 68, 88, 91}	none	{2}	{1}	∅	∅	∅	{2}
{66, 89}	none	{1}	∅	∅	∅	{2}	{6,7}
{66, 69, 89, 92}	none	{1}	∅	∅	∅	{2}	{4,5}
{67, 90}	none	∅	∅	∅	∅	∅	{4}
{68, 91}	S2	{2}	{0}	∅	∅	{0}	∅
{69, 92}	S1	{0}	∅	∅	{0}	{2,3}	{4,6}
{70, 93}	none	{4,5}	{2}	∅	∅	∅	{2}
{71, 94}	none	{5-9,11}	{3-8}	{2,3}	∅	∅	∅
{72, 95}	none	∅	∅	∅	{1}	{2}	{3,4,5,7}
{73, 96}	S1	{0}	∅	∅	{0}	{2}	{4}
{74, 97}	none	{4,6}	{2,3}	∅	∅	∅	{2}
{75, 98}	S2	∅	{0}	∅	∅	{0}	{4}
{76}	none	∅	∅	∅	∅	{1}	{3,4}
{77}	none	{4}	{1}	∅	∅	∅	{1}
{78}	S2	∅	{0}	∅	∅	{0}	{4}
{79}	S2	∅	{0}	∅	∅	{0}	{3,5}
{80}	S2	∅	{0}	∅	∅	{0}	{5}
{81}	S3	{6,8}	{4,5}	{0}	∅	∅	{0}
{82}	none	∅	∅	∅	{1}	{3,4}	{5-8}
{83}	S2	∅	{0}	∅	∅	{0}	∅
{84}	S2	∅	{0}	∅	∅	{0}	{3}
{85}	none	∅	{1}	∅	∅	∅	{3}
{86}	none	{5,6,8}	{3,4,5}	{1}	∅	∅	∅
{99, 108}	none	∅	∅	∅	∅	{1}	{5}
{99, 104, 108}	none	∅	∅	∅	∅	{1}	∅
{100, 105, 109}	none	∅	∅	∅	∅	∅	{4}
{101, 110}	none	{2}	∅	∅	∅	{1}	{3}
{101, 106, 110}	none	{2}	∅	∅	∅	∅	{4}
{102, 111}	none	∅	∅	∅	∅	∅	∅
{103, 112}	none	{3}	{2}	∅	∅	∅	{1}
{103, 107, 112}	none	{3}	{2}	∅	∅	∅	∅
{104}	none	∅	∅	∅	∅	∅	{2}
{105}	none	∅	∅	∅	∅	∅	∅
{106}	none	{2}	∅	∅	∅	{1}	{5,6}

{107}	none	{3}	{2}	∅	∅	∅	{1}
{113}	none	∅	∅	∅	∅	{1}	{5}
{114}	none	∅	∅	∅	∅	∅	∅
{115}	none	∅	∅	∅	∅	∅	∅
{116}	none	{1}	∅	∅	∅	{1}	{3,5}
{117}	none	{5,6}	{3}	∅	∅	∅	{1}
{118}	none	∅	∅	∅	∅	∅	{3}
{119}	none	∅	∅	∅	∅	∅	∅
{120}	none	{1}	∅	∅	∅	{1}	{3}
{121}	S1	{0}	∅	∅	{0}	{1,3}	{5-8}
{122}	none	∅	∅	∅	∅	∅	{3}
{123}	S2	∅	{0}	∅	∅	{0}	∅
{124}	none	{5,6}	{3}	∅	∅	∅	∅
{125}	none	∅	∅	∅	∅	{1}	{2,4,6}
{126}	none	{5,7}	{3,4}	∅	∅	∅	{1}
{127}	S2	∅	{0}	∅	∅	{0}	{4}

Table 26: The number of IRR's of length `n` derived by repeated applications of `F` from a single IRR matching $\{73, 96\}.5$ of length 3 subject to the Search Condition.

IRR outline	n: 3	4	5	6	7	8	9	10
$\{73, 96\}$	1	0	1	0	2	0	4	0
$\{21, 129\}$	0	1	0	2	0	4	0	8
$\{69, 92\}$	0	0	1	0	2	0	4	0
$\{82\}$	0	1	0	2	0	4	0	8
$\{29, 42, 51\}$	0	1	0	2	0	4	0	8
$\{12\}$	0	1	0	2	0	4	0	8
$\{121\}$	0	1	0	2	0	4	0	8
Total	1	5	2	10	4	20	8	40

Table 27: The number of IRR's derived by repeated applications of `F` starting from all the IRR(3) of type LR and LL, in ascending order of difficulty, subject to the Search Condition.

Starting IRR outline	n: 3	4	5	6	7	8	9	10
$\{1, 2, 3\}.4$	3	0	0	0	0	0	0	0
$\{5, 6, 8\}.3$	3	0	0	0	0	0	0	0
$\{9\}.5$	2	0	0	0	0	0	0	0
$\{17\}.2 \times 3$	3	0	0	0	0	0	0	0
$\{20, 128\}.5 \times 2$	2	0	0	0	0	0	0	0

{99,108}.4	1	0	0	0	0	0	0	0
{102,111}.4	1	0	0	0	0	0	0	0
{113}.2	1	0	0	0	0	0	0	0
{114}.2	1	0	0	0	0	0	0	0
{115}.2	1	0	0	0	0	0	0	0
{119}.5 $\times$ 2	2	0	0	0	0	0	0	0
{73,96}.5,{66,89}	2	10	4	20	8	40	16	80
{66,89}								

checked to here ***************

These results can be abbreviated by deleting column 3 and combining entries with the same value of column 2. While doing this it was noticed that the column 1 entries with non-empty intersection often give the same column 2 entry. In Table 28 these sets of lines were replaced by a single line in which the column 1 entry is the union of the separate column 1 entries. These unions were also present in Table irr sets. There was also the case $\{101, 106, 110\} \xrightarrow{\mathtt{F}} \{66, 69, 89, 92\}$ which was deemed to include $\{106\} \xrightarrow{\mathtt{F}} \{66, 89\}$. This extra detail is only retained in Table irr sets. Likewise for $\{107\} \xrightarrow{\mathtt{F}} \{65, 88\}$. The interpretation of Table 28 is that any IRR matching any member of column 1 or a proper subset thereof under `F` gives an IRR matching any corresponding member of column 2 or a proper subset thereof. For this to work, each set in column 2 must be replaced by the superset, which is one of the unions above, that it is a subset of.

Table 28: Relations between sets of IRR outlines of types LL and LR under F

Original set of IRR outlines	Set of sets of IRR outlines derived by F
{1}-{12},{17}-{19},{29, 42, 51},{30, 43, 52},	∅
{35},{39},{64, 67, 87, 90},{72, 95},{76},{82}	∅
{99, 104, 108},{100, 105, 109},{113},{114}	∅
{118},{119}	∅
{13, 57, 61}	{18}, {79}, {39}, {116}
{14, 58, 62}	{17}, {80}, {40}, {113}
{15, 59},{16, 60, 63}	{19}, {81}, {41}, {117}
{20, 128}	{72, 95}
{21, 129},{22, 130}	{73, 96}
{23, 131}	{74, 97}
{24, 132},{25, 133}	{75, 98}
{26, 48}	{14, 58}, {76}, {35}, {3}, {101, 110}
{27, 49}	{13, 57}, {78}, {36}, {1}, {99, 108}
{28, 50}	{16, 60}, {77}, {37}, {3}, {103, 112}
{31, 44, 53}	{14, 58, 62}, {76}, {35}, {3, 5}, {101, 106, 110}
{32, 45, 54}	{13, 57, 61}, {78}, {36}, {1, 4}, {99, 104, 108}
{33, 46, 55}	{13, 57, 61}, {78}, {36}, {1, 4}, {100, 105, 109}
{34, 47, 56}	{16, 60, 63}, {77}, {37}, {3, 5}, {103, 107, 112}
{36}, {40}	{62}, {5}, {106}
{37}	{61}, {4}, {104}
{38}	{61}, {4}, {105}
{41}	{63}, {5}, {107}
{61}	{18}, {79}, {39}, {116}
{62}	{17}, {80}, {40}, {113}
{63}	{19}, {81}, {41}, {117}
{65, 68, 88, 91}	{22, 130}, {84}, {30, 43, 52}, {11}, {120}
{66, 69, 89, 92},{73, 96}	{21, 129}, {82}, {29, 42, 51}, {12}, {121}
{70, 93}	{24, 132}, {85}, {33, 46, 55}, {10}, {122}
{71, 94}	{23, 131}, {86}, {34, 47, 56}, {12}, {124}
{74, 97}	{20, 128}, {85}, {31, 44, 53}, {9}, {118}
{75, 98}	{25, 133}, {83}, {32, 45, 54}, {12}, {123}
{77},{79},{83},{84}	{125}, {26, 48}, {7}
{78},{80},{85}	{127}, {27, 49}, {8}
{81},{86}	{126}, {28, 50}, {6}
{101, 106, 110}	{66, 69, 89, 92}
{102, 111}	{64, 67, 87, 90}
{103, 107, 112}	{65, 68, 88, 91}
{115},{122}	{67, 90}
{116},{120},{121}	{69, 92}
{117},{124}	{70, 93}

$\{\mathtt{123}\}$	$\{\mathtt{68,91}\}$
$\{\mathtt{125}\}$	$\{\mathtt{72,95}\}$
$\{\mathtt{126}\}$	$\{\mathtt{74,97}\}$
$\{\mathtt{127}\}$	$\{\mathtt{75,98}\}$

Notice that $\{\mathtt{3,5}\}$ has arisen as a distinct set of IRR outlines that does not appear in the LHS's of Table irr sets. Because both $\{\mathtt{3}\}$ and $\{\mathtt{5}\}$ lead to no new outlines under $\mathtt{F}$, this should not be a problem. In this table, many members of column 1 are a subset of other members of column 1. This should not be a problem in the analysis. The frequencies of these types of IRR will be shown separately in the subsequent analysis. These phenomena seem to be closely related with the fact that many IRR triplet outlines give under $\mathtt{F}$ the same set of IRR triplet outlines e.g. ... because they occur with often the same original IRR outlines. These sets of IRR outlines need to be identified.

Also cases were found where there were multiple derivations of an IRR outline under $\mathtt{F}$ starting from a set of IRR outlines representing a single IRR. In these cases, the origin has an abbreviated form appearing multiple times.(e.g.). Because in these case each such origin appears only once, in the frequency analysis these cases will be combined e.g.

Many pairs of IRR outlines have the same set of derived IRR outlines. This explains why most occurrences of a subset of IRR outlines give the same result under $\mathtt{F}$ as the full set of IRR outlines. This results from some IRR having a subset of the origins of other ones, and the LHS and RHS the same. Also some IRR have origins matching the outline multiple times.

Although this table is rather long, regularities make it easy to generate for example if IRR($\mathtt{n}$) has an origin of the form $\mathtt{2}\ldots\mathtt{x}\underline{\mathtt{y}}$ then from (2) and the symbol $\mathtt{y}$, the origins of the corresponding IRR($\mathtt{n}+\mathtt{1}$) can be written down immediately. Likewise for the LHS of IRR($\mathtt{n}+\mathtt{1}$), so only the RHS's require lookups in (1),(4), or in the list of results starting from ex3 to the end of Section 1. The RHS's can often be copied from a previous set. The above sort order makes it easy to ensure closure i.e. every IRR($\mathtt{n}+\mathtt{1}$) outline of type LR or RL appears as an IRR($\mathtt{n}$) outline. In most cases this involved just adding in any LHS states that were in the corresponding LHS of the IRR($\mathtt{n}+\mathtt{1}$) but not yet in the LHS of the IRR($\mathtt{n}$). Sometimes extra IRR($\mathtt{n}$) outlines were needed, needing a new set (they were numbered afterwards).

Derived from this is the following, Table 29, that shows all the relationships between the triplet outlines involved in Table 22. Triplet outlines not of extendable type (LR or RL) i.e. where the pointer does not show in the RHS, are ignored. In Table 29, as before, multiple triplet numbers on the left indicate that each one of these leads to all the triplet numbers on the right. The symbol $\mathtt{x}$ (states of the LHS) after the set number and dot takes all possible values available in the LHS of the relations. When only one value is possible it is indicated.

Table 29: Triplet relations derived from Table 22

$$
\begin{array}{rcl}
\{\mathsf{25.x}, \mathsf{28.x}\} & \Rightarrow & \{\mathsf{49.x}, \mathsf{3.x}, \mathsf{82.x}\} \\
\{\mathsf{26.x}, \mathsf{29.x}, \mathsf{30.x}\} & \Rightarrow & \{\mathsf{48.x}, \mathsf{1.x}\} \\
\{\mathsf{27.x}, \mathsf{31.3}\} & \Rightarrow & \{\mathsf{51.x}, \mathsf{3.x}, \mathsf{84.x}\} \\
\{\mathsf{32.x}, \mathsf{35.x}, \mathsf{37.x}\} & \Rightarrow & \{\mathsf{53.x}, \mathsf{5.x}, \mathsf{85.x}\} \\
\{\mathsf{33.x}, \mathsf{34.4}, \mathsf{38.x}, \mathsf{39.x}\} & \Rightarrow & \{\mathsf{52.x}, \mathsf{4.x}\} \\
\{\mathsf{36.x}, \mathsf{40.3}\} & \Rightarrow & \{\mathsf{54.x}, \mathsf{5.x}, \mathsf{86.x}\} \\
\mathsf{41.x} & \Rightarrow & \mathsf{13.x} \\
\{\mathsf{42.x}, \mathsf{45.x}, \mathsf{46.x}\} & \Rightarrow & \{\mathsf{12.x}, \mathsf{65.x}, \mathsf{32.x}\} \\
\{\mathsf{43.x}, \mathsf{47.3}\} & \Rightarrow & \{\mathsf{15.x}, \mathsf{64.x}, \mathsf{33.x}, \mathsf{89.x}\} \\
\mathsf{44.x} & \Rightarrow & \{\mathsf{13.x}, \mathsf{87.x}\} \\
\{\mathsf{48.x}, \mathsf{52.x}\} & \Rightarrow & \{\mathsf{17.x}, \mathsf{66.x}, \mathsf{91.x}\} \\
\{\mathsf{49.x}, \mathsf{53.x}\} & \Rightarrow & \{\mathsf{16.x}, \mathsf{67.x}, \mathsf{35.x}\} \\
\{\mathsf{50.4}, \mathsf{51.x}, \mathsf{54.x}\} & \Rightarrow & \{\mathsf{18.x}, \mathsf{68.x}, \mathsf{36.x}, \mathsf{92.x}\} \\
\{\mathsf{55.x}, \mathsf{57.x}\} & \Rightarrow & \{\mathsf{21.x}, \mathsf{70.x}, \mathsf{93.x}\} \\
\{\mathsf{56.x}, \mathsf{58.x}, \mathsf{61.x}\} & \Rightarrow & \{\mathsf{20.x}, \mathsf{94.x}\} \\
\mathsf{59.x} & \Rightarrow & \{\mathsf{23.x}, \mathsf{71.x}, \mathsf{39.x}, \mathsf{95.x}\} \\
\mathsf{60.3} & \Rightarrow & \{\mathsf{22.3}, \mathsf{72.3}, \mathsf{40.3}, \mathsf{97.3}\} \\
\mathsf{62.x} & \Rightarrow & \{\mathsf{19.x}, \mathsf{71.x}, \mathsf{37.x}\} \\
\mathsf{63.x} & \Rightarrow & \{\mathsf{24.x}, \mathsf{69.x}, \mathsf{38.x}, \mathsf{96.x}\} \\
\{\mathsf{64.x}, \mathsf{66.x}, \mathsf{69.x}, \mathsf{70.x}\} & \Rightarrow & \{\mathsf{98.x}, \mathsf{25.x}, \mathsf{41.x}, \mathsf{6.x}\} \\
\{\mathsf{65.x}, \mathsf{67.x}, \mathsf{71.x}\} & \Rightarrow & \{\mathsf{100.x}, \mathsf{26.x}, \mathsf{42.x}, \mathsf{7.x}\} \\
\{\mathsf{68.x}, \mathsf{72.3}\} & \Rightarrow & \{\mathsf{99.x}, \mathsf{27.x}, \mathsf{43.x}\} \\
\{\mathsf{73.x}, \mathsf{75.x}\} & \Rightarrow & \{\mathsf{103.x}, \mathsf{10.x}\} \\
\{\mathsf{74.x}, \mathsf{76.x}, \mathsf{79.x}\} & \Rightarrow & \{\mathsf{102.x}, \mathsf{11.x}\} \\
\mathsf{77.x} & \Rightarrow & \{\mathsf{105.x}, \mathsf{30.x}, \mathsf{46.x}, \mathsf{9.x}\} \\
\mathsf{78.3} & \Rightarrow & \{\mathsf{104.3}, \mathsf{31.3}, \mathsf{47.3}, \mathsf{11.3}\} \\
\mathsf{80.x} & \Rightarrow & \{\mathsf{101.x}, \mathsf{28.x}, \mathsf{44.x}, \mathsf{8.x}\} \\
\mathsf{81.x} & \Rightarrow & \{\mathsf{106.x}, \mathsf{29.x}, \mathsf{45.x}, \mathsf{11.x}\} \\
\{\mathsf{82.x}, \mathsf{85.x}\} & \Rightarrow & \{\mathsf{56.x}, \mathsf{74.x}\} \\
\{\mathsf{84.x}, \mathsf{86.x}\} & \Rightarrow & \{\mathsf{55.x}, \mathsf{73.x}\} \\
\{\mathsf{87.x}, \mathsf{91.x}, \mathsf{93.x}, \mathsf{94.x}\} & \Rightarrow & \{\mathsf{58.x}, \mathsf{76.x}\} \\
\{\mathsf{89.x}, \mathsf{96.x}\} & \Rightarrow & \{\mathsf{57.x}, \mathsf{75.x}\} \\
\{\mathsf{92.x}, \mathsf{97.3}\} & \Rightarrow & \{\mathsf{59.x}, \mathsf{77.x}\} \\
\{\mathsf{99.x}, \mathsf{104.3}\} & \Rightarrow & \{\mathsf{62.x}, \mathsf{80.x}\} \\
\{\mathsf{100.x}, \mathsf{105.x}, \mathsf{106.x}\} & \Rightarrow & \{\mathsf{63.x}, \mathsf{81.x}\} \\
\{\mathsf{102.x}, \mathsf{103.x}\} & \Rightarrow & \{\mathsf{61.x}, \mathsf{79.x}\} \\
\{\mathsf{1.x} - \mathsf{24.x}, \mathsf{83.4}, \mathsf{88.4}, \mathsf{90.2}, \mathsf{95.x}, \mathsf{98.x}, \mathsf{101.x}\} & \Rightarrow & \emptyset
\end{array}
$$

Not all the RHS's of these relations are disjoint. Numbers $\mathsf{3}, \mathsf{5}, \mathsf{13}$ and $\mathsf{71}$ occur twice and $\mathsf{11}$ occurs 3 times and the following are missing from the RHS's $\mathsf{2}, \mathsf{14}, \mathsf{34}, \mathsf{50}, \mathsf{60}, \mathsf{78}, \mathsf{83}, \mathsf{88},$ and $\mathsf{90}$.

The following cycles (having many overlaps and divided into 3 disjoint groups)

$$
\begin{array}{l}
\mathtt{79} \Leftrightarrow \mathtt{102} \\
\mathtt{58} \Leftrightarrow \mathtt{94} \\
\mathtt{76} \Rightarrow \mathtt{102} \Rightarrow \mathtt{61} \Rightarrow \mathtt{94} \Rightarrow \mathtt{76} \\
\\
\mathtt{54} \Leftrightarrow \mathtt{36} \\
\mathtt{51} \Rightarrow \mathtt{68} \Rightarrow \mathtt{27} \Rightarrow \mathtt{51} \\
\mathtt{54} \Rightarrow \mathtt{68} \Rightarrow \mathtt{27} \Rightarrow \mathtt{51} \Rightarrow \mathtt{36} \Rightarrow \mathtt{54} \\
\\
\mathtt{35} \Leftrightarrow \mathtt{53} \\
\mathtt{65} \Leftrightarrow \mathtt{42} \\
\mathtt{106} \Leftrightarrow \mathtt{81} \\
\mathtt{45} \Rightarrow \mathtt{65} \Rightarrow \mathtt{100} \Rightarrow \mathtt{81} \Rightarrow \mathtt{45} \\
\mathtt{32} \Rightarrow \mathtt{53} \Rightarrow \mathtt{67} \Rightarrow \mathtt{42} \Rightarrow \mathtt{32} \\
\mathtt{25} \Rightarrow \mathtt{49} \Rightarrow \mathtt{67} \Rightarrow \mathtt{100} \Rightarrow \mathtt{63} \Rightarrow \mathtt{69} \Rightarrow \mathtt{25} \\
\mathtt{49} \Rightarrow \mathtt{35} \Rightarrow \mathtt{53} \Rightarrow \mathtt{67} \Rightarrow \mathtt{100} \Rightarrow \mathtt{63} \Rightarrow \mathtt{38} \Rightarrow \mathtt{52} \Rightarrow \mathtt{66} \Rightarrow \mathtt{25} \Rightarrow \mathtt{49} \\
\mathtt{63} \Rightarrow \mathtt{38} \Rightarrow \mathtt{52} \Rightarrow \mathtt{66} \Rightarrow \mathtt{25} \Rightarrow \mathtt{49} \Rightarrow \mathtt{67} \Rightarrow \mathtt{100} \Rightarrow \mathtt{63} \\
\mathtt{65} \Rightarrow \mathtt{100} \Rightarrow \mathtt{63} \Rightarrow \mathtt{69} \Rightarrow \mathtt{25} \Rightarrow \mathtt{49} \Rightarrow \mathtt{67} \Rightarrow \mathtt{42} \Rightarrow \mathtt{65} \\
\mathtt{100} \Rightarrow \mathtt{81} \Rightarrow \mathtt{45} \Rightarrow \mathtt{32} \Rightarrow \mathtt{53} \Rightarrow \mathtt{67} \Rightarrow \mathtt{100} \\
\mathtt{49} \Rightarrow \mathtt{67} \Rightarrow \mathtt{100} \Rightarrow \mathtt{81} \Rightarrow \mathtt{29} \Rightarrow \mathtt{48} \Rightarrow \mathtt{66} \Rightarrow \mathtt{25} \Rightarrow \mathtt{49}
\end{array}
\tag{686}
$$

were found in 29 by attempting to draw out the complete graph and noticing when a new edge (connection) is added whether or not any new cycles were obvious. The importance of cycles is that they show that there are an infinite number of IRR and allow a recursive definition to be made which defines an infinite subset of the IRR. Searching for cycles was done more systematically leading to Table 30. This lists (1) each set (node) in Table 29,(2) the strongly connected component (SCC) it is in ($\in \mathtt{S} = \{\mathtt{C1}, \mathtt{C2}, \mathtt{C3}\}$) except for those SCC's that have just a single set (single node), (3-5) whether or not there is a path from the starting node to each member of `S` in a search that ends when a node (except the starting node) found is in `S` or is a terminating node,(6) "order" which represents the order of derivation. This is the length of the longest path from the starting node to any node in `S` or a terminating node.

Table 30 can be verified by first establishing that all pairs of sets in `C1` connect to each other (directly or indirectly) in both directions. Likewise for `C2`, and `C3`. This can be done by first picking out all relations with both members in the same member of `S` and chaining these together. Thus they are subsets of SCC's. Then the other columns of Table 30 can be verified easily in any order such that "order" is non-decreasing. Doing this will show that every path from any member of `S` to another one is one of

$$
\begin{array}{l}
\mathtt{C2} \rightarrow \mathtt{C1} \\
\mathtt{C2} \rightarrow \mathtt{C3} \\
\mathtt{C3} \rightarrow \mathtt{C1}
\end{array}
.
\tag{687}
$$

Therefore for each set in `C1`, there is no path to a set outside `C1` and then back into it. If there was such a path all the intermediate sets would have to be included in `C1`. Likewise for `C2` and `C3`. This shows that members of `S` are maximal that is they are indeed SCC's. Table 30 could presumably be obtained by an extension of Tarjan's algorithm for obtaining the SCC's from a directed graph.

Table 30: Resolution of the directed graph defined by Table 29

set	SCC	→ `C1`	→ `C2`	→ `C3`	order
1 − 24	none	✗	✗	✗	0
25	C3	✓	✗	✓	3
26	C3	✗	✗	✓	1
27	C2	✓	✓	✓	4
28	none	✓	✗	✓	3
29	C3	✗	✗	✓	1
30	none	✗	✗	✓	1
31	none	✓	✓	✓	4
32	C3	✓	✗	✓	3
33	none	✗	✗	✓	1
34	none	✗	✗	✓	1
35	C3	✓	✗	✓	3
36	C2	✓	✓	✓	4
37	none	✓	✗	✓	3
38	C3	✗	✗	✓	1
39	none	✗	✗	✓	1
40	none	✓	✓	✓	4
41	none	✗	✗	✗	1
42	C3	✗	✗	✓	1
43	none	✓	✗	✓	4
44	none	✓	✗	✗	2
45	C3	✗	✗	✓	1
46	none	✗	✗	✓	1
47	none	✓	✗	✓	4
48	C3	✓	✗	✓	2
49	C3	✗	✗	✓	1
50	none	✗	✓	✓	4
51	C2	✗	✓	✓	4
52	C3	✓	✗	✓	2
53	C3	✗	✗	✓	1
54	C2	✗	✓	✓	4
55	none	✓	✗	✓	2
56	none	✓	✗	✗	1
57	C3	✓	✗	✓	2

58	C1	✓	✗	✗	1
59	none	✗	✗	✓	2
60	none	✓	✓	✓	7
61	C1	✓	✗	✗	1
62	none	✓	✗	✓	4
63	C3	✗	✗	✓	1
64	none	✗	✗	✓	2
65	C3	✗	✗	✓	1
66	C3	✗	✗	✓	2
67	C3	✗	✗	✓	1
68	C2	✓	✓	✓	6
69	C3	✗	✗	✓	2
70	C3	✗	✗	✓	2
71	none	✗	✗	✓	1
72	none	✓	✓	✓	6
73	none	✓	✗	✗	2
74	none	✓	✗	✗	1
75	none	✓	✗	✗	2
76	C1	✓	✗	✗	1
77	none	✗	✗	✓	2
78	none	✓	✓	✓	6
79	C1	✓	✗	✗	1
80	none	✓	✗	✓	4
81	C3	✗	✗	✓	1
82	none	✓	✗	✗	2
83	none	✗	✗	✗	0
84	none	✓	✗	✓	3
85	none	✓	✗	✗	2
86	none	✓	✗	✓	3
87	none	✓	✗	✗	1
88	none	✗	✗	✗	0
89	none	✓	✗	✓	3
90	none	✗	✗	✗	0
91	none	✓	✗	✗	1
92	none	✗	✗	✓	3
93	none	✓	✗	✗	1
94	C1	✓	✗	✗	1
95	none	✗	✗	✗	0
96	C3	✓	✗	✓	3
97	none	✗	✗	✓	3
98	none	✗	✗	✗	0
99	none	✓	✗	✓	5
100	C3	✗	✗	✓	1
101	none	✗	✗	✗	0

102	C1	✓	✗	✗	1
103	none	✓	✗	✗	1
104	none	✓	✗	✓	5
105	none	✗	✗	✓	1
106	C3	✗	✗	✓	1

****

11.2 Using the directed graph of relations amongst IRR outlines to count derived IRR's

Table 31: Members of IRR(3) of types LL and LR and their corresponding sets of IRR outlines from Table 22

IRR(3) member	set of IRR outlines	multiplicity
(40)	{17}.2	3
(41)	{80}.2	3
(42)	{40}.2	3
(43).a	{114}.2	1
(43).c	{113}.2	1
(43).d	{115}.2	1
(44)	{66, 89}.2	3
(48)	{126}.3	1
(49)	{28, 50}.3	1
(50)	{6}.3	1
(51)	{127}.3	1
(52)	{27, 49}.3	1
(53)	{8}.3	1
(54).1	{63}.3	1
(54).2	{5}.3	1
(54).3	{107}.3	1
(55)	{70, 93}.3	1
(56)	{71, 94}.3	1
(64).b	{16, 60}.4	1
(64).c	{13, 57}.4	1
(64).e	{15, 59}.4	1
(65).b	{77}.4	1
(65){.c, .e}	{78}.4	2
(66).b	{37}.4	1
(66).c	{36}.4	1
(66).e	{38}.4	1
(67).b	{3}.4	1
(67).c	{1}.4	1

(67).e	{2}.4	1
(68).b	{103, 112}.4	1
(68).c	{99, 108}.4	1
(68).e	{102, 111}.4	1
(69)	{75, 98}.4	1
(70)	{73, 96}.4	1
(76)	{20, 128}.5	2
(77)	{85}.5	2
(78)	{31, 44, 53}.5	2
(79)	{9}.5	2
(80)	{119}.5	2

The following is the list of sets obtained from 22 which are the starting points for the generation of the IRR.

$$\begin{array}{l}\texttt{1.4, 2.4, 3.4, 5.3, 7.3, 8.5, 12.4, 14.4, 15.4, 16.2, 19.5, 26.3, 27.3,}\\ \texttt{28.5, 32.4, 33.4, 34.4, 35.2, 37.5, 42.3, 43.3, 44.5, 48.4, 50.4, 51.4,}\\ \texttt{54.3, 56.2, 59.3, 60.3, 61.4, 63.4, 64.4, 65.4, 67.2, 71.5, 74.2, 77.3,}\\ \texttt{78.3, 79.4, 81.4, 83.4, 84.4, 86.3, 88.4, 89.4, 90.2, 99.3, 100.3, 101.5}\end{array} \tag{688}$$

The purpose of deriving Table 22, the relations derived from it Table 29 and the resolution of it into SCC's and SCC reachability relations Table res, is to be able to use the latter to define recursively an infinite subset of the IRR, and ultimately do this for all the IRR. To show how this can be done, consider first just SCC `C1` and those IRR that can be obtained starting from those matching sets 79.4 and 61.4 in `C1`. First note that from Theorem 2.2, if IRR outlines `X` and `Y` satisfy the relation `X` ⇒ `Y` and if two distinct IRR `r1` and `r2` match `X` then distinct derived IRR, say `s1` and `s2` will match `Y`. Likewise if `X1` and `X2` are distinct outlines matching distinct IRR `r1` and `r2` respectively, and `X1` ⇒ `Y` and `X2` ⇒ `Y` then there are two distinct IRR say `s1` and `s2` derived from `r1` and `r2` respectively which match `Y`. Now it should be clear how this can be used to count the IRR derived from the set 79.4 in IRR(3) as in the following table.

Table 32: Counting the IRR derived from set `79.4`

total#	`n`	`58.4`	`61.4`	`76.4`	`79.4`	`94.4`	`102.4`
1	3	0	0	0	1	0	0
1	4	0	0	0	0	0	1
2	5	0	1	0	1	0	0
2	6	0	0	0	0	1	1
4	7	1	1	1	1	0	0
4	8	0	0	0	0	2	2
8	9	2	2	2	2	0	0
...							

from which it is clear that the numbers now double for an increase of `n` by 2. When the calculation is done similarly for `C2` it is a little more complicated because there are cycles of length 2,3, and 5 giving

Table 33: Counting the IRR derived from set `27.3`

total#	n	27.3	36.3	51.3	54.3	68.3
1	3	1	0	0	0	0
1	4	0	0	1	0	0
2	5	0	1	0	0	1
2	6	1	0	0	0	1
3	7	0	1	1	0	1
4	8	1	1	0	1	1
5	9	1	1	1	1	1
7	10	1	2	1	1	2
9	11	2	2	1	2	2
12	12	2	3	2	2	3
16	13	3	4	2	3	4
...						

Likewise this could be done starting from each of the sets in (688) noticing that in most of these cases the initial sets are not in an SCC so paths to the SCC's must be first traced out.

12 old stuff

To make things simpler, in the first instance, the IRR outlines were further abbreviated thus incorporating more IRR in each outline IRR, by deleting the state of the LHS and all the RHS. Duplicates of IRR outlines were eliminated thus shortening the table. The table was then checked for closure i.e. every outline IRR($\mathtt{n}+\mathtt{1}$) must also appear as an outline IRR($\mathtt{n}$) and the argument would have been repeated to generate the IRR($\mathtt{n}+\mathtt{1}$) from these etc. until closure was obtained. In this case however the procedure quickly came to an end. This is indicated by the smallest `n` value being 4 for only sets `14` and `19`. Each of these outlines IRR($\mathtt{n}$) appeared as IRR($\mathtt{n}+\mathtt{1}$) but were not obtained by abbreviating entries in Table 1.

This was done first for those results with the arbitrary symbols ... on the right, giving Table 34. The meaning of Table 34 is as follows: for each IRR($\mathtt{n}$) outline, if there is an IRR matching it of length `n`, all the IRR($\mathtt{n}+\mathtt{1}$) derived from it as above must match one of the set of IRR outlines corresponding to it in the right hand part of the table.

Table 34: Recursive definition of a superset of some of the IRR

set number	smallest	IRR(n)		IRR(n + 1)	
	n	Origin	LHS	Origin	LHS
1	3	1da...	bc...	2dd...	ab...
				2ad...	cb...
				2cd...	db...
2	3	1dc...	bd...	2dd...	ab...
				2ad...	cb...
				2cd...	db...
3	3	1dd...	ba...	2dd...	ab...
				2ad...	cb...
				2cd...	db...
4	3	1ea...	bc...	2de...	ab...
				2ae...	cb...
				2ce...	db...
5	3	1ec...	bd...	2de...	ab...
				2ae...	cb...
				2ce...	db...
6	3	1ed...	ba...	2de...	ab...
				2ae...	cb...
				2ce...	db...
7	3	2ad...	cb...	1da...	bc...
				1ea...	bc...
8	3	2ae...	cb...	1da...	bc...
				1ea...	bc...
9	3	2cd...	db...	1dc...	bd...
				1ec...	bd...
10	3	2ce...	db...	1dc...	bd...
				1ec...	bd...
11	3	2dd...	ab...	1dd...	ba...
				1ed...	ba...
12	3	2de...	ab...	1dd...	ba...
				1ed...	ba...
13	3	3ab...	cb...	5ca...	ac...
				5ea...	ac...
				3ea...	bc...
				4ca...	cc...
14	4	3ac...	cc...	5ca...	ac...
				5ea...	ac...
				3ea...	bc...
				4ca...	cc...
15	3	3ae...	ca...	5ca...	ac...
				5ea...	ac...
				3ea...	bc...

				4$\underline{\texttt{c}}$a...	cc...
16	3	3$\underline{\texttt{e}}$a...	bc...	5$\underline{\texttt{c}}$e... 5$\underline{\texttt{e}}$e... 3$\underline{\texttt{e}}$e... 4$\underline{\texttt{c}}$e...	ab... ab... bb... cb...
17	3	3$\underline{\texttt{e}}$e...	bb...	5$\underline{\texttt{c}}$e... 5$\underline{\texttt{e}}$e... 3$\underline{\texttt{e}}$e... 4$\underline{\texttt{c}}$e...	ab... ab... bb... cb...
18	3	4$\underline{\texttt{b}}$b...	bb...	4$\underline{\texttt{b}}$b... 3$\underline{\texttt{a}}$b...	bb... cb...
19	4	4$\underline{\texttt{b}}$c...	bc...	4$\underline{\texttt{b}}$b... 3$\underline{\texttt{a}}$b...	bb... cb...
20	3	4$\underline{\texttt{b}}$e...	ba...	4$\underline{\texttt{b}}$b... 3$\underline{\texttt{a}}$b...	bb... cb...
21	3	4$\underline{\texttt{c}}$a...	cc...	4$\underline{\texttt{b}}$c... 3$\underline{\texttt{a}}$c...	bc... cc...
22	3	4$\underline{\texttt{c}}$e...	cb...	4$\underline{\texttt{b}}$c... 3$\underline{\texttt{a}}$c...	bc... cc...
23	3	4$\underline{\texttt{e}}$c...	aa...	4$\underline{\texttt{b}}$e... 3$\underline{\texttt{a}}$e...	ba... ca...
24	3	4$\underline{\texttt{e}}$e...	aa...	4$\underline{\texttt{b}}$e... 3$\underline{\texttt{a}}$e...	ba... ca...
25	3	5$\underline{\texttt{c}}$a...	ac...	4$\underline{\texttt{e}}$c...	aa...
26	3	5$\underline{\texttt{c}}$e...	ab...	4$\underline{\texttt{e}}$c...	aa...
27	3	5$\underline{\texttt{e}}$a...	ac...	4$\underline{\texttt{e}}$e...	aa...
28	3	5$\underline{\texttt{e}}$e...	ab...	4$\underline{\texttt{e}}$e...	aa...

Thus an infinite number of IRR can be captured by a recursive description of this type, and the exact set captured is determined by the initial set of known IRR that each match one of the patterns. Note that not all rules matching the patterns are necessarily IRR.

One consequence of this last stage of abbreviation of the IRR outlines is that the information in the RHS has been lost, in particular the type of the IRR i.e. determining whether or not it is extendable to an IRR one symbol longer. Thus in Table 34 some IRR outlines can be vacuous i.e. have no matching IRR. This happens when an IRR($\texttt{n}$) outline has no matching IRR that are of extendable types RL or LR. Further work with the full IRR outlines above will attempt to refine this.

Apart from this, because of the restriction on the direction of the movement of the pointer in the derivation of origins, in general not all the IRR are obtained like this. This restriction will be systematically overcome.

Table 35: Analysis of the simultaneous induction defined by Table 34

Initial conditions	Implication statements	results
17($\mathtt{n} = 3$)	17 $\Rightarrow$ 17	17($\mathtt{n} \geq 3$)
22($\mathtt{n} = 3$)	17 $\Rightarrow$ 22	22($\mathtt{n} \geq 3$)
2($\mathtt{n} = 3$) and 9($\mathtt{n} = 3$)	9 $\Leftrightarrow$ 2	2($\mathtt{n} \geq 3$) and 9($\mathtt{n} \geq 3$)
5($\mathtt{n} = 3$) and 10($\mathtt{n} = 3$)	5 $\Leftrightarrow$ 10	5($\mathtt{n} \geq 3$) and 10($\mathtt{n} \geq 3$)
7($\mathtt{n} = 3$)	2 $\Rightarrow$ 7	7($\mathtt{n} \geq 3$)
12($\mathtt{n} = 3$)	5 $\Rightarrow$ 12	12($\mathtt{n} \geq 3$)
8($\mathtt{n} = 3$)	5 $\Rightarrow$ 8	8($\mathtt{n} \geq 3$)
1($\mathtt{n} = 3$)	7 $\Rightarrow$ 1	1($\mathtt{n} \geq 3$)
4($\mathtt{n} = 3$)	7 $\Rightarrow$ 4	4($\mathtt{n} \geq 3$)
3($\mathtt{n} = 3$)	12 $\Rightarrow$ 3	3($\mathtt{n} \geq 3$)
6($\mathtt{n} = 3$)	12 $\Rightarrow$ 6	6($\mathtt{n} \geq 3$)
11($\mathtt{n} = 3$)	1 $\Rightarrow$ 11	11($\mathtt{n} \geq 3$)
14($\mathtt{n} = 4$) and 21($\mathtt{n} = 3$)	14 $\Leftrightarrow$ 21	14($\mathtt{n}$ even and $\mathtt{n} \geq 4$)
		21($\mathtt{n}$ odd and $\mathtt{n} \geq 3$)
	21 $\Rightarrow$ 19	19($\mathtt{n}$ even and $\mathtt{n} \geq 4$)
13($\mathtt{n} = 3$)	19 $\Rightarrow$ 13	13($\mathtt{n}$ odd and $\mathtt{n} \geq 3$)
18($\mathtt{n} = 3$)	19 $\Rightarrow$ 18	18($\mathtt{n}$ odd and $\mathtt{n} \geq 3$)
13($\mathtt{n} = 3$)	18 $\Rightarrow$ 13	13($\mathtt{n}$ even and $\mathtt{n} \geq 4$)
		13($\mathtt{n} \geq 3$)
16($\mathtt{n} = 3$)	13 $\Rightarrow$ 16	16($\mathtt{n} \geq 3$)
28($\mathtt{n} = 3$)	16 $\Rightarrow$ 28	28($\mathtt{n} \geq 3$)
24($\mathtt{n} = 3$)	28 $\Rightarrow$ 24	24($\mathtt{n} \geq 3$)
26($\mathtt{n} = 3$)	16 $\Rightarrow$ 26	26($\mathtt{n} \geq 3$)
23($\mathtt{n} = 3$)	26 $\Rightarrow$ 23	23($\mathtt{n} \geq 3$)
15($\mathtt{n} = 3$)	23 $\Rightarrow$ 15	15($\mathtt{n} \geq 3$)
21($\mathtt{n} = 3$)	13 $\Rightarrow$ 21	21($\mathtt{n} \geq 3$)
25($\mathtt{n} = 3$)	13 $\Rightarrow$ 25	25($\mathtt{n} \geq 3$)
27($\mathtt{n} = 3$)	13 $\Rightarrow$ 27	27($\mathtt{n} \geq 3$)
19($\mathtt{n} = 4$)	21 $\Rightarrow$ 19	19($\mathtt{n} \geq 4$)
20($\mathtt{n} = 3$)	24 $\Rightarrow$ 20	20($\mathtt{n} \geq 3$)
18($\mathtt{n} = 3$)	20 $\Rightarrow$ 18	18($\mathtt{n} \geq 3$)
14($\mathtt{n} = 4$)	21 $\Rightarrow$ 14	14($\mathtt{n} \geq 4$)

To avoid repeated use of the phrase “is/are true for”, the statement number is given followed by the conditions on `n` in parentheses. In Table 35 the sets in Table 34 are referred to by number (using monospaced typewriter font and not to be confused with the symbols), so for example the first set will be written as 1 $\Rightarrow$ 11 , 1 $\Rightarrow$ 7 and 1 $\Rightarrow$ 9. That `n` is increased by one on the RHS will be implicit because this is always the case. In this way a directed graph can be constructed from Table 34 indicating the implication statements one for each row. The implication statement is “if there is an IRR `r` of length n of type RL or LR matching the column IRR($\mathtt{n}$) then there is an IRR $\mathtt{r}'$ of length $\mathtt{n}+\mathtt{1}$ derived from $\mathtt{r}$ matching each of the outlines corresponding to it in column IRR($\mathtt{n}+\mathtt{1}$). Because the type of $\mathtt{r}'$ is not indicated in Table 34, the closure procedure will in general yield some outlines not corresponding to any IRR.

The point of the closure procedure was to attempt the simultaneous proof by induction that determines the set of values of $\mathtt{n}$ that apply to each set in Table 34. Table 35 is listed in the order in which the derivation can proceed. It was obtained by first searching for cycles in the directed graph, first shorter ones then longer ones, because they allow induction arguments to be made. Then many implication statements allow other consequent statements to be made.

By noting the state of the LHS in each set of IRR outlines in Table 34, which is the same for the IRR($\mathtt{n}$) and derived IRR($\mathtt{n}+\mathtt{1}$ outlines), it is possible to find all possible states of the LHS's for each set number by carrying them forward to the derived IRR outlines for next value of $\mathtt{n}$. It turns out that statements $\mathtt{1}-\mathtt{12}$ can have the state of the LHS as $\mathtt{1}$ or $\mathtt{2}$, and statements $\mathtt{13}-\mathtt{28}$ can have the state of the LHS $\in\{\mathtt{3},\mathtt{4},\mathtt{5}\}$. This argument should be extended to establish the set of values of $\mathtt{n}$ and the RHS's for each of these states.

The analogous computations will now be carried out starting from the subset of Table 1 with ... on the left in each CS before returning to using the original IRR outlines, and taking account of unexpected pointer movement in the derivation of origins.

12.1 Repeating the analysis for the entries of Table 1 with the unknown symbols on the left

Here the arbitrary symbols on the left, giving Table 36.

Table 36: Recursive definition of a subset of the IRR of type LR

set number	IRR($\mathtt{n}$) Origin	LHS	IRR($\mathtt{n}+\mathtt{1}$) Origin	LHS
1	$\mathtt{1\ldots b\underline{a}}$	$\mathtt{\ldots cd}$	$\emptyset$	
2	$\mathtt{1\ldots e\underline{a}}$	$\mathtt{\ldots cd}$	$\emptyset$	
3	$\mathtt{1\ldots a\underline{b}}$	$\mathtt{\ldots bd}$	$\emptyset$	
4	$\mathtt{1\ldots d\underline{b}}$	$\mathtt{\ldots cd}$	$\emptyset$	
5	$\mathtt{1\ldots b\underline{c}}$	$\mathtt{\ldots ca}$	$\emptyset$	
6	$\mathtt{1\ldots c\underline{c}}$	$\mathtt{\ldots aa}$	$\emptyset$	
7	$\mathtt{1\ldots d\underline{c}}$	$\mathtt{\ldots ca}$	$\emptyset$	
8	$\mathtt{2\ldots a\underline{b}}$	$\mathtt{\ldots bc}$	$\mathtt{3\ldots b\underline{c}}$	$\mathtt{\ldots ca}$
			$\mathtt{1\ldots b\underline{a}}$	$\mathtt{\ldots cd}$
			$\mathtt{5\ldots b\underline{a}}$	$\mathtt{\ldots ce}$

9	2...d$\underline{b}$ ...cc	3...b$\underline{c}$ ...ca
		1...b$\underline{a}$ ...cd
		5...b$\underline{a}$ ...ce
10	2...b$\underline{e}$ ...cc	3...e$\underline{c}$ ...ca
		1...e$\underline{a}$ ...cd
		5...e$\underline{a}$ ...ce
11	2...c$\underline{e}$ ...ac	3...e$\underline{c}$ ...ca
		1...e$\underline{a}$ ...cd
		5...e$\underline{a}$ ...ce
12	2...d$\underline{e}$ ...cc	3...e$\underline{c}$ ...ca
		1...e$\underline{a}$ ...cd
		5...e$\underline{a}$ ...ce
13	3...a$\underline{b}$ ...bc	1...b$\underline{c}$ ...ca
		4...b$\underline{a}$ ...cb
		2...b$\underline{e}$ ...cc
		5...b$\underline{b}$ ...ce
14	3...d$\underline{b}$ ...cc	1...b$\underline{c}$ ...ca
		4...b$\underline{a}$ ...cb
		2...b$\underline{e}$ ...cc
		5...b$\underline{b}$ ...ce
15	3...b$\underline{c}$ ...ca	1...c$\underline{c}$ ...aa
		4...c$\underline{a}$ ...ab
		2...c$\underline{e}$ ...ac
		5...c$\underline{b}$ ...ae
16	3...e$\underline{c}$ ...ca	1...c$\underline{c}$ ...aa
		4...c$\underline{a}$ ...ab
		2...c$\underline{e}$ ...ac
		5...c$\underline{b}$ ...ae
17	3...a$\underline{d}$ ...ec	1...d$\underline{c}$ ...ca
		4...d$\underline{a}$ ...cb
		2...d$\underline{e}$ ...cc
		5...d$\underline{b}$ ...ce
18	3...b$\underline{d}$ ...ec	1...d$\underline{c}$ ...ca
		4...d$\underline{a}$ ...cb
		2...d$\underline{e}$ ...cc
		5...d$\underline{b}$ ...ce
19	3...d$\underline{d}$ ...ac	1...d$\underline{c}$ ...ca
		4...d$\underline{a}$ ...cb
		2...d$\underline{e}$ ...cc
		5...d$\underline{b}$ ...ce
20	4...b$\underline{a}$ ...cb	5...a$\underline{d}$ ...ba
		2...a$\underline{b}$ ...bc
		3...a$\underline{b}$ ...bc

			1...a$\underline{b}$	...bd
21	4...c$\underline{a}$	...ab	5...a$\underline{d}$	...ba
			2...a$\underline{b}$	...bc
			3...a$\underline{b}$	...bc
			1...a$\underline{b}$	...bd
22	4...d$\underline{a}$	...cb	5...a$\underline{d}$	...ba
			2...a$\underline{b}$	...bc
			3...a$\underline{b}$	...bc
			1...a$\underline{b}$	...bd
23	4...a$\underline{d}$	...ec	5...d$\underline{d}$	...ca
			2...d$\underline{b}$	...cc
			3...d$\underline{b}$	...cc
			1...d$\underline{b}$	...cd
24	4...b$\underline{d}$	...ec	5...d$\underline{d}$	...ca
			2...d$\underline{b}$	...cc
			3...d$\underline{b}$	...cc
			1...d$\underline{b}$	...cd
25	4...d$\underline{d}$	...ac	5...d$\underline{d}$	...ca
			2...d$\underline{b}$	...cc
			3...d$\underline{b}$	...cc
			1...d$\underline{b}$	...cd
26	5...b$\underline{a}$	...ce	3...a$\underline{d}$	...ec
			4...a$\underline{d}$	...ec
27	5...e$\underline{a}$	...ce	3...a$\underline{d}$	...ec
			4...a$\underline{d}$	...ec
28	5...b$\underline{b}$	...ce	3...b$\underline{d}$	...ec
			4...b$\underline{d}$	...ec
29	5...c$\underline{b}$	...ae	3...b$\underline{d}$	...ec
			4...b$\underline{d}$	...ec
30	5...d$\underline{b}$	...ce	3...b$\underline{d}$	...ec
			4...b$\underline{d}$	...ec
31	5...a$\underline{d}$	...ba	3...d$\underline{d}$	...ac
			4...d$\underline{d}$	...ac
32	5...d$\underline{d}$	...ca	3...d$\underline{d}$	...ac
			4...d$\underline{d}$	...ac

In these results the "smallest n" values were all 3. The analysis of the possible states of the LHS's in the IRR($\mathtt{n}$) and IRR($\mathtt{n}+\mathtt{1}$) shows that they are $\{2, 3, 4, 5\}$ for each set number.

The analysis of the simultaneous induction defined by Table 36 proceeds in a similar way to that in Table 34 giving Table 35. In the notation of Table 35 the following two results are obtained:

$$\begin{array}{ccccc} \mathtt{11}(\mathtt{n}=\mathtt{3}) & \mathtt{16}(\mathtt{n}=\mathtt{3}) & \mathtt{11} \Leftrightarrow \mathtt{16} & \mathtt{11}(\mathtt{n}\geq\mathtt{3}) & \mathtt{16}(\mathtt{n}\geq\mathtt{3}) \\ \mathtt{13}(\mathtt{n}=\mathtt{3}) & \mathtt{20}(\mathtt{n}=\mathtt{3}) & \mathtt{13} \Leftrightarrow \mathtt{20} & \mathtt{13}(\mathtt{n}\geq\mathtt{3}) & \mathtt{20}(\mathtt{n}\geq\mathtt{3}) \end{array} \tag{689}$$

All the remaining results are of the form $\mathtt{x}(\mathtt{n}=\mathtt{3})$ and $\mathtt{y}(\mathtt{n}) \Rightarrow \mathtt{x}(\mathtt{n}+\mathtt{1})$ (written as $\mathtt{y} \Rightarrow \mathtt{x}$ for brevity) implies $\mathtt{x}(\mathtt{n}\geq\mathtt{3})$ where $\mathtt{y}$ has already been proved for $\mathtt{n}\geq\mathtt{3}$. To indicate them it is sufficient to write all the implication statements in the order that they are used (reading across then down).

$$\begin{array}{cccc} 13 \Rightarrow 10 & 10 \Rightarrow 27 & 27 \Rightarrow 17 & 17 \Rightarrow 30 \\ 16 \Rightarrow 29 & 20 \Rightarrow 8 & 8 \Rightarrow 15 & 15 \Rightarrow 21 \\ 21 \Rightarrow 31 & 31 \Rightarrow 19 & 19 \Rightarrow 12 & 19 \Rightarrow 22 \\ 31 \Rightarrow 25 & 25 \Rightarrow 14 & 8 \Rightarrow 26 & 26 \Rightarrow 23 \\ 13 \Rightarrow 28 & 28 \Rightarrow 18 & 28 \Rightarrow 24 & 24 \Rightarrow 32 \\ 23 \Rightarrow 9 & 8 \Rightarrow 1 & 10 \Rightarrow 2 & 22 \Rightarrow 3 \\ 23 \Rightarrow 4 & 13 \Rightarrow 5 & 15 \Rightarrow 6 & 17 \Rightarrow 7 \end{array} \tag{690}$$

The result of this is that each set is included for all $\mathtt{n}\geq\mathtt{3}$.

****

Hence the following addition to the inductive hypothesis (Table 37) is suggested in analogy with Table 34 and Table 36. In Table 37 in the last column, the match to IRR($\mathtt{n}+\mathtt{1}$) is with the appropriate table i.e. Table 34 if the ... is on the right, and Table 36 otherwise. As in Tables 34 and 36, the closure procedure was applied to ensure that all entries in column IRR($\mathtt{n}+\mathtt{1}$) also appear in column IRR($\mathtt{n}$) and all other columns were completed. This only gave two extra rows indicated by the "smallest n" values being 5.

Table 37: Further derived IRR outlines

set number	smallest n	IRR(n) Origin	LHS	IRR(n + 1) Origin	LHS	Matching IRR outlines from Table 34 or 36
1	4	1$\underline{\mathtt{d}}$aa...	bdb...	2$\underline{\mathtt{d}}$da...	abd...	11
				2$\underline{\mathtt{a}}$da...	cbd...	7
				2$\underline{\mathtt{c}}$da...	dbd...	9
2	4	1$\underline{\mathtt{d}}$ab...	bdb...	2$\underline{\mathtt{d}}$da...	abd...	11
				2$\underline{\mathtt{a}}$da...	cbd...	7
				2$\underline{\mathtt{c}}$da...	dbd...	9
3	4	1$\underline{\mathtt{d}}$ca...	bab...	2$\underline{\mathtt{d}}$dc...	aba...	11
				2$\underline{\mathtt{a}}$dc...	cba...	7
				2$\underline{\mathtt{c}}$dc...	dba...	9
4	4	1$\underline{\mathtt{e}}$aa...	bdb...	2$\underline{\mathtt{d}}$ea...	abd...	12
				2$\underline{\mathtt{a}}$ea...	cbd...	8
				2$\underline{\mathtt{c}}$ea...	dbd...	10

5	4	1eab . . . bdb . . .	2dea . . . abd . . .	12
			2aea . . . cbd . . .	8
			2cea . . . dbd . . .	10
6	4	1eca . . . bab . . .	2dec . . . aba . . .	12
			2aec . . . cba . . .	8
			2cec . . . dba . . .	10
7	4	3acb . . . cab . . .	5cac . . . aca . . .	25
			5eac . . . aca . . .	27
			3eac . . . bca . . .	16
			4cac . . . cca . . .	21
8	4	3acc . . . cab . . .	5cac . . . aca . . .	25
			5eac . . . aca . . .	27
			3eac . . . bca . . .	16
			4cac . . . cca . . .	21
9	4	3acd . . . cdb . . .	5cac . . . acd . . .	25
			5eac . . . acd . . .	27
			3eac . . . bcd . . .	16
			4cac . . . ccd . . .	21
10	4	3aeb . . . cca . . .	5cae . . . acc . . .	25
			5eae . . . acc . . .	27
			3eae . . . bcc . . .	16
			4cae . . . ccc . . .	21
			4beb . . . bcc . . .	
			3aeb . . . ccc . . .	
11	5	3aeb . . . ccc . . .	5cae . . . acc . . .	25
			5eae . . . acc . . .	27
			3eae . . . bcc . . .	16
			4cae . . . ccc . . .	21
			4beb . . . bcc . . .	
			3aeb . . . ccc . . .	
12	4	3ead . . . bdb . . .	5cea . . . abd . . .	26
			5eea . . . abd . . .	28
			3eea . . . bbd . . .	17
			4cea . . . cbd . . .	22
13	4	4bcb . . . bab . . .	4bbc . . . bba . . .	18
			3abc . . . cba . . .	13
			2ddb . . . aba . . .	11
			2deb . . . aba . . .	12
			5ceb . . . aba . . .	26
			5eeb . . . aba . . .	28
			3eeb . . . bba . . .	17
			2adb . . . cba . . .	7
			2aeb . . . cba . . .	8

			4$\underline{\texttt{c}}$eb... cba...	22
			2$\underline{\texttt{c}}$db... dba...	9
			2$\underline{\texttt{c}}$eb... dba...	10
14	4	4$\underline{\texttt{b}}$cc... bab...	4$\underline{\texttt{b}}$bc... bba...	18
			3$\underline{\texttt{a}}$bc... cba...	13
			2$\underline{\texttt{d}}$db... aba...	11
			2$\underline{\texttt{d}}$eb... aba...	12
			5$\underline{\texttt{c}}$eb... aba...	26
			5$\underline{\texttt{e}}$eb... aba...	28
			3$\underline{\texttt{e}}$eb... bba...	17
			4$\underline{\texttt{c}}$eb... cba...	22
			2$\underline{\texttt{c}}$db... dba...	9
			2$\underline{\texttt{c}}$eb... dba...	10
15	4	4$\underline{\texttt{b}}$cd... bdb...	4$\underline{\texttt{b}}$bc... bbd...	18
			3$\underline{\texttt{a}}$bc... cbd...	13
			2$\underline{\texttt{d}}$db... abd...	11
			2$\underline{\texttt{d}}$eb... abd...	12
			5$\underline{\texttt{c}}$eb... abd...	26
			5$\underline{\texttt{e}}$eb... abd...	28
			3$\underline{\texttt{e}}$eb... bbd...	17
			2$\underline{\texttt{a}}$db... cbd...	7
			2$\underline{\texttt{a}}$eb... cbd...	8
			4$\underline{\texttt{c}}$eb... cbd...	22
			2$\underline{\texttt{c}}$db... dbd...	9
			2$\underline{\texttt{c}}$eb... dbd...	10
16	4	4$\underline{\texttt{b}}$eb... bca...	4$\underline{\texttt{b}}$be... bbc...	18
			3$\underline{\texttt{a}}$be... cbc...	13
17	5	4$\underline{\texttt{b}}$eb... bcc...	4$\underline{\texttt{b}}$be... bbc...	18
			3$\underline{\texttt{a}}$be... cbc...	13
18	4	4$\underline{\texttt{c}}$ad... cdb...	4$\underline{\texttt{b}}$ca... bcd...	19
			3$\underline{\texttt{a}}$ca... ccd...	14
19	4	4$\underline{\texttt{e}}$cc... adb...	4$\underline{\texttt{b}}$ec... bad...	20
			3$\underline{\texttt{a}}$ec... cad...	15
20	4	4$\underline{\texttt{e}}$ec... adb...	4$\underline{\texttt{b}}$ee... bad...	20
			3$\underline{\texttt{a}}$ee... cad...	15
21	4	5$\underline{\texttt{c}}$ad... adb...	4$\underline{\texttt{e}}$ca... aad...	23
22	4	5$\underline{\texttt{e}}$ad... adb...	4$\underline{\texttt{e}}$ea... aad...	24
23	4	1...ee$\underline{\texttt{a}}$ ...cad	$\emptyset$	
24	4	3...ee$\underline{\texttt{c}}$ ...caa	1...ec$\underline{\texttt{c}}$ ...aaa	6
			4...ec$\underline{\texttt{a}}$ ...aab	21
			2...ec$\underline{\texttt{e}}$ ...aac	11
			5...ec$\underline{\texttt{b}}$ ...aae	29
25	4	3...bd$\underline{\texttt{d}}$ ...cbc	1...dd$\underline{\texttt{c}}$ ...bca	7

				4...dd$\underline{\text{a}}$	...bcb	22
				2...dd$\underline{\text{e}}$	...bcc	12
				5...dd$\underline{\text{b}}$	...bce	30
26	4	4...bd$\underline{\text{d}}$	...cbc	5...dd$\underline{\text{d}}$	...bca	32
				2...dd$\underline{\text{b}}$	...bcc	9
				3...dd$\underline{\text{b}}$	...bcc	14
				1...dd$\underline{\text{b}}$	...bcd	4
27	4	5...ee$\underline{\text{a}}$	...cae	3...ea$\underline{\text{d}}$	...aec	17
				4...ea$\underline{\text{d}}$	...aec	23

The induction cannot be carried out as before and is made unnecessary because in almost all cases, matches were obtained to for the IRR($\mathtt{n}+\mathtt{1}$) in previous tables. In this case the "smallest $\mathtt{n}$" values are actually the only $\mathtt{n}$ values. In the derivation of Table 37 on a few occasions, again the pointer went in the unexpected direction resulting in the following partial reverse rules and finally two members of IRR(5) of type RR. These are not extendable, so this argument ends here:

$$\mathtt{4}\alpha\underline{\mathtt{b}}\mathtt{cbe} \dashleftarrow \begin{cases} \mathtt{4}\underline{\mathtt{b}}\mathtt{bbae} \\ \mathtt{3}\underline{\mathtt{a}}\mathtt{bbae} \end{cases} \tag{691}$$

$$\mathtt{4}\alpha\underline{\mathtt{b}}\mathtt{cce} \dashleftarrow \mathtt{5}\alpha\mathtt{bbe}\underline{\mathtt{a}} \tag{692}$$

$$\mathtt{4}\alpha\underline{\mathtt{b}}\mathtt{cde} \dashleftarrow \emptyset \tag{693}$$

$$\mathtt{3}\alpha\underline{\mathtt{a}}\mathtt{cde} \dashleftarrow \emptyset \tag{694}$$

$$\mathtt{4}\underline{\mathtt{b}}\mathtt{bbae} \to \mathtt{2bbab}\underline{\mathtt{e}} \to \mathtt{4bcaac}\_ \tag{695}$$

$$\mathtt{3}\underline{\mathtt{a}}\mathtt{bbae} \to \mathtt{2cbab}\underline{\mathtt{e}} \to \mathtt{1ababd}\_ \tag{696}$$

The result of this is the set of IRR outlines in Tables 34 36 and 37 that together will match any IRR from the Turing machine.